A Concise Treatise

Reprint of updated First Chapter of the Volume

QUANTUM MECHANICS IN PHASE SPACE

An Overview with Selected Papers

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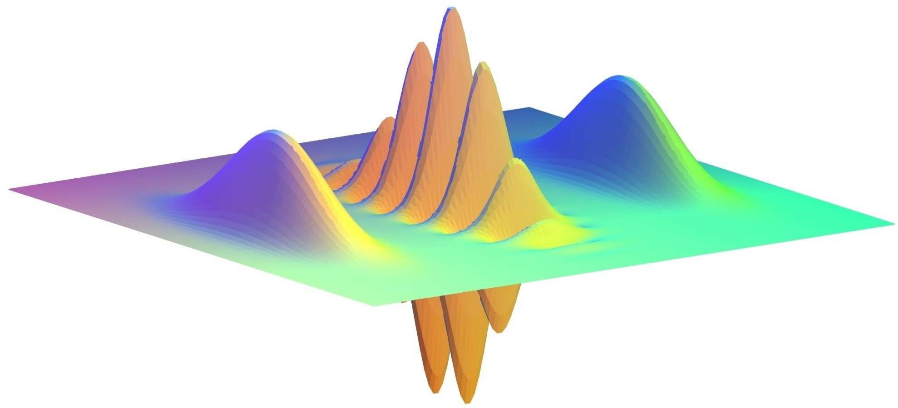
World Scientific Series in 20th Century Physics — Vol. 34
560pp ISBN 981-238-384-0 978-981-238-384-6

http://www.worldscibooks.com/physics/5287.html

ANL-HEP-PR-04-049

Quantum Mechanics in Phase Space



Zachos, Fairlie, & Curtright

CONTENTS

[Contents taken care of by the publisher:]

PREFACE

Wigner's quasi-probability distribution function in phase-space is a special (Weyl-Wigner) representation of the density matrix. It has been useful in describing transport in quantum optics; nuclear physics; and quantum computing, decoherence, and chaos. It is also of importance in signal processing, and the mathematics of algebraic deformation. A remarkable aspect of its internal logic, pioneered by Groenewold and Moyal, has only emerged in the last quarter-century: It furnishes a third, alternative, formulation of quantum mechanics, independent of the conventional Hilbert space, or path integral formulations.

In this logically complete and self-standing formulation, one need not choose sides between coordinate or momentum space. It works in full phase-space, accommodating the uncertainty principle; and it offers unique insights into the classical limit of quantum theory: The variables (observables) in this formulation are c-number functions in phase space instead of operators, with the same interpretation as their classical counterparts, but are composed together in novel algebraic ways.

This volume is a selection of 25 useful papers in the phase-space formulation, with an introductory overview which provides a trail-map to these papers, and an extensive bibliography. (Still, the bibliography makes no pretense to exhaustiveness. An up-to-date database on the large literature of the field, with special emphasis on its mathematical and technical aspects, may be found in http://idefix.physik.uni-freiburg.de/∼star/en/download.html .)

The overview collects often-used formulas and simple illustrations, suitable for applications to a broad range of physics problems, as well as teaching. As a concise treatise, it provides supplementary material which may be used for a beginning graduate course in quantum mechanics.

D Morrissey is thanked for helpful comments, and T Curtright expresses his obligation to Ms Diaz-Heimer.

Errata and other updates to the book may be found on-line at
http://server.physics.miami.edu/∼curtright/QMPS

C. K. Zachos, D. B. Fairlie, and T. L. Curtright

OVERVIEW OF PHASE-SPACE QUANTIZATION

0.1 Introduction

There are at least three logically autonomous alternative paths to quantization. The first is the standard one utilizing operators in Hilbert space, developed by Heisenberg, Schrödinger, Dirac, and others in the 1920s. The second one relies on path integrals, and was conceived by Dirac[Dir33] and constructed by Feynman.

The third one (the bronze medal!) is the phase-space formulation surveyed in this book. It is based on Wigner's (1932) quasi-distribution function[Wig32] and Weyl's (1927) correspondence[Wey27] between ordinary c-number functions in phase space and quantum-mechanical operators in Hilbert space.

The crucial quantum-mechanical composition structure of all such functions, which relies on the $\star$-product, was fully understood by Groenewold (1946)[Gro46], who, together with Moyal (1949)[Moy49], pulled the entire formulation together. Still, insights on interpretation and a full appreciation of its conceptual autonomy took some time to mature with the work of, among others, Takabayasi[Tak54], Baker[Bak58], and Fairlie[Fai64].

This complete formulation is based on the Wigner function (WF), which is a quasi-probability distribution function in phase-space,

$$f(x,p) = \frac{1}{2\pi}\int dy \;\; \psi^*\left(x - \frac{\hbar}{2}y\right) \; e^{-iyp}\psi\left(x + \frac{\hbar}{2}y\right). \tag{1}$$

It is a generating function for all spatial autocorrelation functions of a given quantum-mechanical wave-function $\psi(x)$. More importantly, it is a special representation of the density matrix (in the Weyl correspondence, as detailed in Section 0.12).

Alternatively, in a $2n$-dimensional phase space, it amounts to

$$f(x,p) = \frac{1}{(2\pi\hbar)^n}\int d^n y \;\left\langle x + \frac{y}{2}\right| \boldsymbol{\rho} \left| x - \frac{y}{2}\right\rangle \; e^{-ip\cdot y/\hbar}, \tag{2}$$

where $\psi(x) = \langle x|\psi\rangle$ in the density operator $\boldsymbol{\rho}$,

$$\boldsymbol{\rho} \;= \int d^n z \int d^n x d^n p \;\left| x + \frac{z}{2}\right\rangle f(x,p)\; e^{ip\cdot z/\hbar} \left\langle x - \frac{z}{2}\right| . \tag{3}$$

There are several outstanding reviews on the subject: refs[HOS84,Tak89,Ber80,BJ84,Lit86], [deA98,Shi79,Tat83,Coh95,KN91,Kub64,DeG74,KW90], [Ber77,Lee95,Dah01,Sch02,DHS00,CZ83,Gad95,HH02], [Str57,McD88,Leo97,Sny80,Bal75,BFF78].

Nevertheless, the central conceit of the present overview is that the above input wave-functions may ultimately be bypassed, since the WFs are determined, in principle, as the solutions of suitable functional equations in phase space. Connections to the Hilbert space operator formulation of quantum mechanics may thus be ignored, in principle—even though they are provided in Section 0.12 for pedagogy and confirmation of the formulation's equivalence. One might then envision an imaginary world in which this formulation of quantum mechanics had preceded the conventional Hilbert-space formulation, and its own techniques and methods had arisen independently, perhaps out of generalizations of classical mechanics and statistical mechanics.

It is not only wave-functions that are missing in this formulation. Beyond the ubiquitous (noncommutative, associative, pseudodifferential) operation, the $\star$-product, which encodes the entire quantum-mechanical action, there are no linear operators. Expectations of observables and transition amplitudes are phase-space integrals of c-number functions, weighted by the WF, as in statistical mechanics.

Consequently, even though the WF is not positive-semidefinite (it can be, and usually is negative in parts of phase-space [Wig32]), the computation of expectations and the associated concepts are evocative of classical probability theory, as emphasized by Moyal. Still, telltale features of quantum mechanics are reflected in the noncommutative multiplication of such c-number phase-space functions through the $\star$-product, in systematic analogy to operator multiplication in Hilbert space.

This formulation of quantum mechanics is useful in describing *quantum* transport processes in phase space, notably in quantum optics[Sch02,Leo97,SM00]; nuclear and particle physics[Bak60,SP81,WH99,MM84,CC03,BJY04]; condensed matter [DO85,MMP94,DBB02,KKFR89,JG93,BP96,Ram04,KL01,JBM03]; the study of semiclassical limits of mesoscopic systems [Imr67,OR57,Sch69,Ber77,KW87,OM95,MS95,MOT98,Vor89,Vo78,Hel76,Wer95,Ara95,Mah87,Rob93,CdD04,Pul06,Zdn06]; and the transition to classical statistical mechanics[VMdG61,JD99,Fre87,BD98,Dek77,Raj83,HY96,CV98,SM00,FLM98,FZ01,Zal03,CKTM07].

Since observables are expressed by essentially *common variables in both their quantum and classical configurations*, this formulation is the natural language in which to investigate quantum signatures of chaos[HW80,GHSS05,MNV08,CSA09,Haa10] and decoherence[Ber77,JN90,Zu91,ZP94,Hab90,BC99,KZZ02,KJ99,FBA96,Kol96,GH93,CL03,BTU93,Mon94,HP03,OC03,BC09,GB03,MMM11] (of utility in, e.g., quantum computing[BHP02,MPS02,TGS05]).

It likewise provides crucial intuition in quantum-mechanical interference problems[Wis97,Son09], molecular Talbot-Lau interferometry[NH08], probability flows as negative probability backflows[BM94,FMS00,BV90], and measurements of atomic systems[Smi93,Dun95,Lei96,KPM97,Lvo01,JS02,BHS02,Ber02,Cas91].

The intriguing mathematical structure of the formulation is of relevance to Lie Algebras[FFZ89]; martingales in turbulence[Fan03]; and string field theory[BKM03]. It has also been retrofitted into M-theory and quantum field theory advances linked to noncommutative geometry[SW99,Fil96] (for reviews, see [Cas00,Har01,DN01,HS02]), and to matrix models[Tay01,KS02]; these apply spacetime uncertainty principles[Pei33,Yo89,JY98,SST00] reliant on the $\star$-product. (Transverse spatial dimensions act formally as momenta, and, analogously to quantum mechanics, their uncertainty is increased or decreased inversely to the uncertainty of a given direction.)

As a significant aside, in formal emulation of quantum mechanics,[Vill48] the WF has extensive practical applications in signal processing, filtering, and engineering (time-frequency analysis), since, mathematically, time and frequency constitute a pair of Fourier-conjugate variables, just like the $\mathfrak{x}$ and $\mathfrak{p}$ pair of phase space[a].

For simplicity, the formulation will be mostly illustrated here for one coordinate and its conjugate momentum; but generalization to arbitrary-sized phase spaces is straightforward[Bal75,DM86], including infinite-dimensional ones, namely scalar field theory[Dit90,Les84,Na97,CZ99,CPP01,MM94]: the respective WFs are simple products of single-particle WFs.

0.2 The Wigner Function

As already indicated, the quasi-probability measure in phase space is the WF,

$$f(x,p) = \frac{1}{2\pi} \int dy \; \psi^* \left(x - \frac{\hbar}{2} y \right) \; e^{-iyp} \; \psi \left(x + \frac{\hbar}{2} y \right). \tag{4}$$

It is obviously normalized, $\int dpdx f(x,p) = 1$, for normalized input wavefunctions. In the classical limit, $\hbar \to 0$, it would reduce to the probability density in coordinate space, x, usually highly localized, multiplied by δ-functions in momentum: in phase space, the classical limit is "spiky" and certain!

This expression has more $x-p$ symmetry than is apparent, as Fourier transformation to momentum-space wave-functions yields a completely symmetric expression with the roles of x and p reversed, and, upon rescaling of the arguments x and p, a symmetric classical limit.

The WF is also manifestly real[b]. It is further constrained[Bak58] by the Cauchy-Schwarz inequality to be bounded: $-\frac{2}{\hbar} \leq f(x,p) \leq \frac{2}{\hbar}$. Again, this bound disappears in the spiky classical limit. Thus, this quantum-mechanical bound precludes a WF which is a perfectly localized delta function in x and p—the uncertainty principle.

Respectively, p- or x-projection leads to marginal probability densities: a spacelike shadow $\int dp \; f(x,p) = \rho(x)$, *or else* a momentum-space shadow $\int dx f(x,p) = \sigma(p)$. Either is a bona-fide probability density, being positive semidefinite. But these potentialities are actually interwoven. Neither can be conditioned on the other, as the uncertainty principle is fighting back: The WF $f(x,p)$ itself can, and most often is *negative* in some *small* areas of phase-space[Wig32,HOS84,MLD86]. This is illustrated below, and furnishes a hallmark of

[a] Thus, time-varying signals are best represented in a WF as time-varying spectrograms, analogously to a music score, i.e. the changing distribution of frequencies is monitored in time[deB67,BBL80,Wok97,QC96,MH97,Coh95,Gro01,Fla99]: even though the description is constrained and redundant, it furnishes an intuitive picture of the signal which a mere time profile or frequency spectrogram fails to convey.

Applications abound[CGB91,Lou96,MH97] in bioengineering, acoustics, speech analysis, vision processing, radar imaging, turbulence microstructure analysis, seismic imaging[WL10], and the monitoring of internal combustion engine-knocking, failing helicopter-component vibrations, atmospheric radio occultations[GLL10] and so on.

[b] In one space dimension, by virtue of non-degeneracy, ψ has the same effect as ψ^*, and f turns out to be p-even; but this is not a property used here.

QM interference in this language. Such negative features thus serve to monitor quantum coherence; and their attenuation, respectively, its loss. (In fact, the only pure state WF which is non-negative is the Gaussian[Hud74], a state of maximum entropy[Raj83].)

The counter-intuitive "negative probability" aspects of this quasi-probability distribution have been explored and interpreted [Bar45,Fey87,BM94,MLD86] (for a popular review, see [LPM98]). For instance, negative probability flows may be regarded as legitimate probability backflows in interesting settings[BM94]. Nevertheless, the WF for atomic systems can still be measured in the laboratory, albeit indirectly, and reconstructed [Smi93,Dun95,Lei96,KPM97,Lvo01,Lut96,BAD96,BHS02,Ber02,BRWK99,Vog89].

Smoothing f by a filter of size larger than $\hbar$ (e.g., convolving with a phase-space Gaussian) necessarily results in a *positive-semidefinite function*, i.e. it may be thought to have been smeared or blurred to a classical[c] distribution[deB67,Car76,Ste80,OW81,Raj83].

It is thus evident that phase-space patches of uniformly negative value for f cannot be larger than a few $\hbar$, since, otherwise, smoothing by such an $\hbar$-filter would fail to obliterate them as required above. That is, *negative patches are small, a microscopic phenomenon*, in general, in some sense shielded by the uncertainty principle. Monitoring negative WF features and their attenuation in time (as quantum information leaks into the environment) affords a measure of decoherence and drift towards a classical (mixed) state[KJ99].

Among real functions, the WFs comprise a rather small, highly constrained, set. When is a real function $f(x,p)$ a bona-fide, pure-state, Wigner function of the form (4)? Evidently, when its Fourier transform (the cross-spectral density) "left-right" factorizes,

$$\tilde{f}(x,y) = \int dp\; e^{ipy} f(x,p) \;=\; g_L^*(x-\hbar y/2)\; g_R(x+\hbar y/2)\ . \tag{5}$$

That is,

$$\frac{\partial^2 \quad \ln \tilde{f}}{\partial(x-\hbar y/2)\ \partial(x+\hbar y/2)} = 0\ , \tag{6}$$

so that, for real f, $\quad g_L = g_R$.

Nevertheless, as indicated, the WF *is* a distribution function, after all: it provides the integration measure in phase space to yield expectation values of observables from

[c] This one is called the Husimi distribution[Tak89,TA99], and sometimes information scientists examine it preferentially on account of its non-negative feature. Nevertheless, it comes with a substantially heavy price, as it needs to be "dressed" back to the WF, for all practical purposes, when equivalent quantum expectation values are computed with it: i.e., unlike the WF, it does *not* serve as an immediate quasi-probability distribution with no further measure (see Section 0.13). The negative feature of the WF is, in the last analysis, an asset, and not a liability, and provides an efficient description of "beats"[BBL80,Wok97,QC96,MH97,Coh95], cf. Fig. 1.
Caution: If, instead, strictly *inequivalent* expectation values were taken with the Husimi distribution *without* the requisite dressing of Section 0.13, i.e. improperly, as though it were a bona-fide probability distribution, such expectation values would actually reflect *loss of quantum information*: they would represent semi-classically smeared observables[WO87].

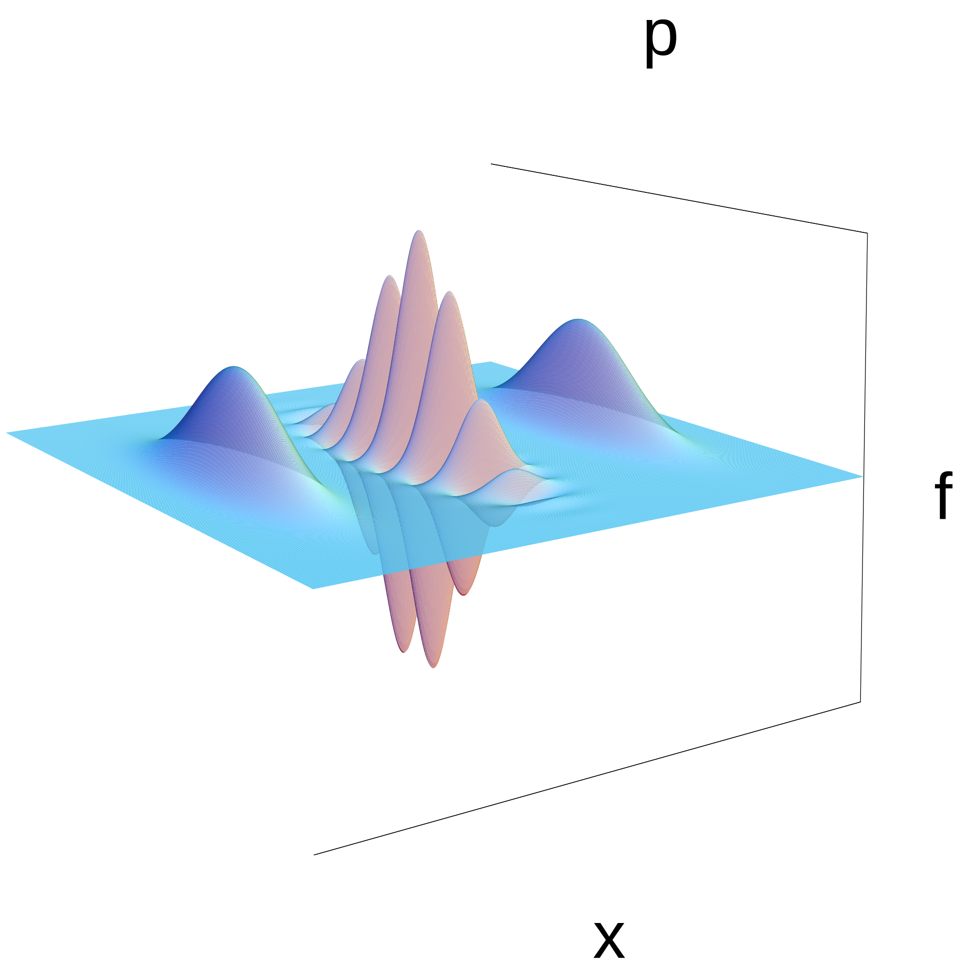


Figure 1. Wigner function of a pair of Gaussian wavepackets centered at $x = \pm a$: $f(x,p;a) = \exp(-(x^2+p^2))(\exp(-a^2)\cosh(2ax) + \cos(2pa))/(\pi(1+e^{-a^2}))$. (For simplicity, $\hbar = 1$ here. The corresponding wave-function is $\psi(x;a) = \left(\exp\left(-(x+a)^2/2\right) + \exp\left(-(x-a)^2/2\right)\right)/(\pi^{1/4}\sqrt{2+2e^{-a^2}})$.) Here, $a = 6$ is chosen, quite larger than the width of the Gaussians. Note the phase-space interference structure ("beats") with negative values in the x region between the two packets where there is no wave-function support—hence vanishing probability for the presence of the particle. The oscillation frequency in the p-direction is a/π. Thus, it increases with growing separation a, ultimately smearing away the interference structure.

corresponding phase-space c-number functions. Such functions are often familiar classical quantities; but, in general, they are uniquely associated to suitably ordered operators through *Weyl's correspondence rule*[Wey27]. Given an operator (in gothic script) ordered in this prescription,

$$\mathfrak{G}(\mathfrak{x},\mathfrak{p}) = \frac{1}{(2\pi)^2}\int d\tau d\sigma dx dp \; g(x,p)\exp(i\tau(\mathfrak{p}-p)+i\sigma(\mathfrak{x}-x)) \ , \tag{7}$$

the corresponding phase-space function $g(x,p)$ (the *Weyl kernel function*, or the *Wigner transform* of that operator) is obtained by

$$\mathfrak{p} \longmapsto p, \qquad\qquad \mathfrak{x} \longmapsto x \ . \tag{8}$$

That operator's expectation value is then given by a "phase-space average" [Gro46,Moy49,Bas48],

$$\langle \mathfrak{G} \rangle = \int dxdp \; f(x,p) \; g(x,p). \tag{9}$$

The kernel function $g(x,p)$ is often the unmodified classical observable expression, such as a conventional Hamiltonian, $H = p^2/2m + V(x)$, i.e. the transition from classical mechanics is straightforward ("quantization").

However, the kernel function contains $\hbar$ corrections when there are quantum-mechanical ordering ambiguities in the observables, such as in the kernel of the square of the angular momentum, $\mathfrak{L}\cdot\mathfrak{L}$. This one contains an additional term $-3\hbar^2/2$ introduced by the Weyl ordering[She59,DS82,DS02], beyond the mere classical expression, L^2. In fact, with suitable averaging, this quantum offset accounts for the nontrivial angular momentum $L=\hbar$ of the ground-state Bohr orbit, when the standard Hydrogen quantum ground state has vanishing $\langle \mathfrak{L}\cdot\mathfrak{L}\rangle = 0$.

In such cases (including momentum-dependent potentials), even nontrivial $O(\hbar)$ quantum corrections in the phase-space kernel functions (which characterize different operator orderings) can be produced efficiently without direct, cumbersome consideration of operators[CZ02,Hie84]. More detailed discussion of the Weyl and alternate correspondence maps is provided in Sections 0.12 and 0.13.

In this sense, expectation values of the physical observables specified by kernel functions $g(x,p)$ are computed through integration with the WF, $f(x,p)$, in close analogy to classical probability theory, except for the non-positive-definiteness of the distribution function. This operation corresponds to tracing an operator with the density matrix (cf. Section 0.12).

0.3 Solving for the Wigner Function

Given a specification of observables, the next step is to find the relevant WF for a given Hamiltonian. Can this be done without solving for the Schrödinger wavefunctions ψ, i.e. not using Schrödinger's equation directly? Indeed, the functional equations which f satisfies completely determine it.

Firstly, its dynamical evolution is specified by *Moyal's equation.* This is the extension of Liouville's theorem of classical mechanics, for a classical Hamiltonian $H(x,p)$, namely $\partial_t f + \{f, H\} = 0$, to quantum mechanics, in this language[Wig32,Bas48,Moy49]:

$$\frac{\partial f}{\partial t} = \frac{H \star f - f \star H}{i\hbar} \equiv \{\!\{H, f\}\!\} \ , \tag{10}$$

where the $\star$-product[Gro46] is

$$\star \equiv e^{\frac{i\hbar}{2}(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x)} \ . \tag{11}$$

The right-hand side of (10) is dubbed the "Moyal Bracket" (MB), and the quantum commutator is its Weyl-correspondent (its Weyl transform). It is the essentially unique one-parameter ($\hbar$) associative deformation (expansion) of the Poisson Brackets (PB) of classical mechanics[Vey75,BFF78,FLS76,Ar83,Fle90,deW83,BCG97,TD97]. Expansion in $\hbar$ around 0 reveals that it consists of the Poisson Bracket corrected by terms $O(\hbar)$.

Moyal's equation equation (10) also evokes Heisenberg's equation of motion for operators (and von Neumann's for the density matrix), except H and f here are ordinary "classical" phase-space functions, and it is the $\star$-product which now enforces noncommutativity. This language, then, makes the link between quantum commutators and Poisson Brackets more transparent.

Since the $\star$-product involves exponentials of derivative operators, it may be evaluated in practice through translation of function arguments ("Bopp shifts"),

Lemma 0.1

$$f(x,p) \star g(x,p) = f\left(x + \frac{i\hbar}{2}\overrightarrow{\partial}_p,\ p - \frac{i\hbar}{2}\overrightarrow{\partial}_x\right)\ g(x,p). \tag{12}$$

The equivalent Fourier representation of the $\star$-product is[Neu31,Bak58]

$$\begin{aligned} f \star g = \frac{1}{\hbar^2\pi^2} \int dp' dp'' dx' dx''\ f(x',p')\ g(x'',p'') \\ \times \exp\left(\frac{-2i}{\hbar}\left(p(x'-x'') + p'(x''-x) + p''(x-x')\right)\right) . \end{aligned} \tag{13}$$

An alternate integral representation of this product is[HOS84]

$$f\star g = (\hbar\pi)^{-2} \int dp' dp'' dx' dx''\ f(x+x', p+p')\ g(x+x'', p+p'')\ \exp\left(\frac{2i}{\hbar}\left(x'p'' - x''p'\right)\right), \tag{14}$$

which readily displays noncommutativity and associativity. □

A fundamental Theorem (0.1) examined later dictates that $\star$-multiplication of c-number phase-space functions is in complete isomorphism to Hilbert-space operator multiplication[Gro46] of the respective Weyl transforms,

$$\mathfrak{A}(\mathfrak{x},\mathfrak{p})\ \mathfrak{B}(\mathfrak{x},\mathfrak{p}) = \frac{1}{(2\pi)^2} \int d\tau d\sigma dx dp\ (a \star b)\ \exp(i\tau(\mathfrak{p}-p) + i\sigma(\mathfrak{x}-x)). \tag{15}$$

The cyclic phase-space trace is directly seen in the representation (14) to reduce to a plain product, if there is *only one* $\star$ involved,

Lemma 0.2

$$\int dpdx \ f \star g = \int dpdx \ fg = \int dpdx \ g \star f. \tag{16}$$

Moyal's equation is necessary, but does not suffice to specify the WF for a system. In the conventional formulation of quantum mechanics, systematic solution of time-dependent equations is usually predicated on the spectrum of stationary ones. Time-independent pure-state Wigner functions $\star$-commute with H; but, clearly, not every function $\star$-commuting with H can be a bona-fide WF (e.g., any $\star$-function of H will $\star$-commute with H).

Static WFs obey even more powerful functional $\star$-genvalue equations[Fai64] (also see [Bas48,Kun67,Coh76,Dah83]),

$$\begin{aligned} H(x,p) \star f(x,p) &= H\left(x + \frac{i\hbar}{2}\vec{\partial}_p \ , \ p - \frac{i\hbar}{2}\vec{\partial}_x\right) \ f(x,p) \\ &= f(x,p) \star H(x,p) = E \ f(x,p) \ , \end{aligned} \tag{17}$$

where E is the energy eigenvalue of $\mathfrak{H}\psi = E\psi$ in Hilbert space. These amount to a complete characterization of the WFs[CFZ98]. (NB. Observe the $\hbar \rightarrow 0$ transition to the classical limit.)

Lemma 0.3 For real functions $f(x,p)$, the Wigner form (4) for pure static eigenstates is equivalent to compliance with the $\star$-genvalue equations (17) ($\Re$ and $\Im$ parts).

Proof

$$\begin{aligned} & H(x,p) \star f(x,p) = \\ &= \frac{1}{2\pi}\left((p - i\frac{\hbar}{2}\vec{\partial}_x)^2/2m + V(x)\right) \int dy \ e^{-iy(p+i\frac{\hbar}{2}\overleftarrow{\partial}_x)} \psi^*(x - \frac{\hbar}{2}y) \ \psi(x + \frac{\hbar}{2}y) \\ &= \frac{1}{2\pi}\int dy \ \left((p - i\frac{\hbar}{2}\vec{\partial}_x)^2/2m + V(x + \frac{\hbar}{2}y)\right) e^{-iyp} \psi^*(x - \frac{\hbar}{2}y) \ \psi(x + \frac{\hbar}{2}y) \\ &= \frac{1}{2\pi}\int dy \ e^{-iyp}\left((i\vec{\partial}_y + i\frac{\hbar}{2}\vec{\partial}_x)^2/2m + V(x + \frac{\hbar}{2}y)\right) \psi^*(x - \frac{\hbar}{2}y) \ \psi(x + \frac{\hbar}{2}y) \\ &= \frac{1}{2\pi}\int dy \ e^{-iyp} \psi^*(x - \frac{\hbar}{2}y) \ E \ \psi(x + \frac{\hbar}{2}y) \\ &= E \ f(x,p). \end{aligned} \tag{18}$$

Action of the effective differential operators on ψ^* turns out to be null.

Symmetrically,

$$\begin{aligned} & f \star H = \\ &\quad = \frac{1}{2\pi}\int dy \ e^{-iyp}\left(-\frac{1}{2m}(\vec{\partial}_y - \frac{\hbar}{2}\vec{\partial}_x)^2 + V(x - \frac{\hbar}{2}y)\right) \psi^*(x - \frac{\hbar}{2}y) \ \psi(x + \frac{\hbar}{2}y) \\ &\quad = E \ f(x,p), \end{aligned} \tag{19}$$

where the action on ψ is now trivial.

Conversely, the pair of $\star$-eigenvalue equations dictate, for $f(x,p) = \int dy\ e^{-iyp} \tilde{f}(x,y)$,

$$\int dy\ e^{-iyp} \left(-\frac{1}{2m} (\vec{\partial}_y \pm \frac{\hbar}{2}\ \vec{\partial}_x)^2 + V(x \pm \frac{\hbar}{2} y) - E \right) \tilde{f}(x,y) = 0. \tag{20}$$

Hence, real solutions of (17) must be of the form
$f = \int dy\ e^{-iyp} \psi^*(x - \frac{\hbar}{2}y) \psi(x + \frac{\hbar}{2}y)/2\pi$, such that $\mathfrak{H}\psi = E\psi$. □

The eqs (17) lead to spectral properties for WFs[Fai64,CFZ98], as in the Hilbert space formulation. For instance, projective orthogonality of the $\star$-genfunctions follows from associativity, which allows evaluation in two alternate groupings:

$$f \star H \star g = E_f\ f \star g = E_g\ f \star g. \tag{21}$$

Thus, for $E_g \neq E_f$, it is necessary that

$$f \star g = 0. \tag{22}$$

Moreover, precluding degeneracy (which can be treated separately), choosing $f = g$ above yields,

$$f \star H \star f = E_f\ f \star f = H \star f \star f, \tag{23}$$

and hence $f \star f$ must be the stargenfunction in question,

$$f \star f \propto f. \tag{24}$$

Pure state fs then $\star$-project onto their space.

In general, the projective property for a pure state can be shown[Tak54,CFZ98],

Lemma 0.4

$$f_a \star f_b = \frac{1}{h}\ \delta_{a,b}\ f_a\ . \tag{25}$$

The normalization matters[Tak54]: despite linearity of the equations, it prevents naive superposition of solutions. (Quantum mechanical interference works differently here, in comportance with conventional density-matrix formalism.) □

By virtue of (16), for different $\star$-genfunctions, the above dictates that

$$\int dpdx\ fg = 0. \tag{26}$$

Consequently, unless there is zero overlap for all such WFs, at least one of the two must go negative someplace to offset the positive overlap [HOS84,Coh95]—an illustration of the salutary feature of negative-valuedness. Here, this feature is *an asset and not a liability.*

Further note that integrating (17) yields the expectation of the energy,

$$\int H(x,p) f(x,p)\ dxdp = E \int f\ dxdp = E. \tag{27}$$

N.B. Likewise[d], integrating the above projective condition yields

$$\int dxdp\ f^2 = \frac{1}{h}\ , \tag{28}$$

which goes to a divergent result in the classical limit, for unit-normalized fs, as the pure-state WFs grow increasingly spiky.

0.4 The Uncertainty Principle

In classical (non-negative) probability distribution theory, expectation values of non-negative functions are likewise non-negative, and thus yield standard *constraint inequalities* for the constituent pieces of such functions, such as, e.g., moments of the variables.

But it was just discussed that, for WFs which go negative for an arbitrary function g, the expectation $\langle |g|^2\rangle$ need not be ≥ 0. This can be easily illustrated by choosing the support of g to lie mostly in those (small) regions of phase-space where the WF f is negative.

Still, such constraints are not lost for WFs. It turns out they are replaced by

Lemma 0.5

$$\langle g^* \star g\rangle \geq 0\ . \tag{29}$$

In Hilbert space operator formalism, this relation would correspond to the positivity of the norm. This expression is non-negative because it involves a real non-negative integrand for a pure state WF satisfying the above projective condition[e],

$$\int dpdx (g^* \star g) f = h \int dxdp (g^* \star g)(f \star f) = h \int dxdp (f \star g^*) \star (g \star f) = h \int dxdp |g \star f|^2 . \tag{30}$$

□

To produce Heisenberg's uncertainty relation[CZ01], one now only need choose

$$g = a + bx + cp\ , \tag{31}$$

for arbitrary complex coefficients a, b, c.

The resulting positive semi-definite quadratic form is then

$$a^*a + b^*b\langle x\star x\rangle + c^*c\langle p\star p\rangle + (a^*b+b^*a)\langle x\rangle + (a^*c+c^*a)\langle p\rangle + c^*b\langle p\star x\rangle + b^*c\langle x\star p\rangle \geq 0\ , \tag{32}$$

for any a, b, c. The eigenvalues of the corresponding matrix are then non-negative, and thus so must be its determinant. Given

$$x \star x = x^2, \qquad p \star p = p^2, \qquad p \star x = px - i\hbar/2\ , \qquad x \star p = px + i\hbar/2\ , \tag{33}$$

[d]This discussion applies to proper WFs, corresponding to pure states' density matrices. E.g., a sum of two WFs similar to a sum of two classical distributions is not a pure state in general, and does not satisfy the condition (6). For such mixed-state generalizations, the *impurity* is[Gro46] $1 - h\langle f\rangle = \int dxdp\ (f - hf^2) \geq 0$, where the inequality is only saturated into an equality for a pure state. For instance, for $w \equiv (f_a + f_b)/2$ with $f_a \star f_b = 0$, the impurity is nonvanishing, $\int dxdp\ (w - hw^2) = 1/2$. A pure state affords a maximum of information, while the impurity is a measure of lack of information[Fan57,Tak54], characteristic of mixed states and decoherence[CSA09,Haa10]—it is the dominant term in the expansion of the quantum entropy around a pure state[Bra94].

[e]Similarly, if f_1 and f_2 are pure state WFs, the transition probability $(|\int dx \psi_1^*(x)\psi_2(x)|^2)$ between the respective states is also non-negative[OW81], manifestly by the same argument[CZ01], providing for a non-negative phase-space overlap, $\int dpdx f_1 f_2 = (2\pi\hbar)^2 \int dxdp\ |f_1 \star f_2|^2 \geq 0$.

and the usual quantum fluctuations

$$(\Delta x)^2 \equiv \langle (x - \langle x \rangle)^2 \rangle, \qquad\qquad (\Delta p)^2 \equiv \langle (p - \langle p \rangle)^2 \rangle, \tag{34}$$

this condition on the 3×3 matrix determinant simply amounts to

$$(\Delta x)^2 \ (\Delta p)^2 \geq \hbar^2/4 + \Big(\langle (x - \langle x \rangle)(p - \langle p \rangle) \rangle \Big)^2, \tag{35}$$

and hence

$$\Delta x \ \Delta p \ \geq \ \frac{\hbar}{2} \ . \tag{36}$$

The $\hbar$ has entered into the moments' constraint through the action of the $\star$-product [CZ01].

More general choices of g likewise lead to diverse expectations' inequalities in phase space; e.g., in 6-dimensional phase space, the uncertainty for $g = a + bL_x + cL_y$ requires $l(l+1) \geq m(m+1)$, and hence $l \geq m$, and so forth[CZ01,CZ02].

For a more extensive formal discussion of moments, cf. ref[NO86].

0.5 Ehrenfest's Theorem

Moyal's equation (10),

$$\frac{\partial f}{\partial t} = \{\!\{ H, f \}\!\} \ , \tag{37}$$

serves to prove Ehrenfest's theorem for expectation values.

For any phase-space function $k(x,p)$ with no explicit time-dependence,

$$\begin{aligned} \frac{d\langle k \rangle}{dt} &= \int dxdp \ \frac{\partial f}{\partial t} k \\ &= \frac{1}{i\hbar} \int dxdp \ (H \star f - f \star H) \star k \\ &= \int dxdp \ f \{\!\{ k, H \}\!\} = \langle \{\!\{ k, H \}\!\} \rangle \ . \end{aligned} \tag{38}$$

(Any convective time-dependence, $\int dxdp \ (\dot{x} \partial_x \ (fk) + \dot{p} \ \partial_p (fk))$, amounts to an ignorable surface term, $\int dxdp \ (\partial_x (\dot{x} f k) + \partial_p (\dot{p} f k))$, by the x, p equations of motion.)

Note the ostensible sign difference between the correspondent to Heisenberg's equation,

$$\frac{dk}{dt} = \{\!\{ k, H \}\!\} \ , \tag{39}$$

and Moyal's equation above. The x, p equations of motion in such a Heisenberg picture, then, reduce to the classical ones of Hamilton, $\dot{x} = \partial_p H$, $\dot{p} = -\partial_x H$.

Moyal [Moy49] stressed that his eponymous quantum evolution equation (10) contrasts to Liouville's theorem for classical phase-space densities,

$$\frac{df_{cl}}{dt} = \frac{\partial f_{cl}}{\partial t} + \dot{x} \ \partial_x f_{cl} + \dot{p} \ \partial_p f_{cl} = 0 \ . \tag{40}$$

Specifically, unlike its classical counterpart, in general, f does not flow like an incompressible fluid in phase space.

For an arbitrary region Ω about some representative point in phase space,

Lemma 0.6

$$\frac{d}{dt}\int_\Omega dxdp\ f = \int_\Omega dxdp\left(\frac{\partial f}{\partial t} + \partial_x(\dot{x}f) + \partial_p(\dot{p}f)\right) = \int_\Omega dxdp\ (\{\!\{H,f\}\!\} - \{H,f\}) \neq 0\ . \quad (41)$$

That is, the phase-space region does not conserve in time the number of points swarming about the representative point: points diffuse away, in general, without maintaining the density of the quantum quasi-probability fluid; and, conversely, they are not prevented from coming together, in contrast to deterministic flow behavior. Still, *for infinite Ω encompassing the entire phase space*, both surface terms above vanish to yield a time-invariant normalization for the WF. □

The $O(\hbar^2)$ higher momentum derivatives of the WF present in the MB (but absent in the PB—higher space derivatives probing nonlinearity in the potential) modify the Liouville flow into characteristic quantum configurations[KZZ02,FBA96,ZP94].

0.6 Illustration: the Harmonic Oscillator

To illustrate the formalism on a simple prototype problem, one may look at the harmonic oscillator. In the spirit of this picture, in fact, one can eschew solving the Schrödinger problem and plugging the wavefunctions into (4). Instead, for $H = (p^2 + x^2)/2$ (with $m = 1$, $\omega = 1$; i.e., with $\sqrt{m\omega}$ absorbed into x and into $1/p$, and $1/\omega$ into H), one may solve (17) directly,

$$\left(\left(x + \frac{i\hbar}{2}\partial_p\right)^2 + \left(p - \frac{i\hbar}{2}\partial_x\right)^2 - 2E\right) f(x,p) = 0. \quad (42)$$

For this Hamiltonian, then, the equation has collapsed to two simple Partial Differential Equations.

The first one, the $\Im$maginary part,

$$(x\partial_p - p\partial_x)f = 0\ , \quad (43)$$

restricts f to depend on only one variable, the scalar in phase space,

$$z \equiv \frac{4}{\hbar}H = \frac{2}{\hbar}(x^2 + p^2)\ .$$

Thus the second one, the $\Re$eal part, is a simple Ordinary Differential Equation,

$$\left(\frac{z}{4} - z\partial_z^2 - \partial_z - \frac{E}{\hbar}\right) f(z) = 0. \quad (44)$$

Setting $f(z) = \exp(-z/2)L(z)$ yields Laguerre's equation,

$$\left(z\partial_z^2 + (1-z)\partial_z + \frac{E}{\hbar} - \frac{1}{2}\right) L(z) = 0. \quad (45)$$

It is solved by Laguerre polynomials,

$$L_n = \frac{1}{n!} e^z \; \partial_z^n (e^{-z} z^n) \; , \tag{46}$$

for $n = E/\hbar - 1/2 = 0, 1, 2, ...$, so that the $\star$-gen-Wigner-functions are[Gro46]

$$f_n = \frac{(-1)^n}{\pi \hbar} \; e^{-2H/\hbar} \; L_n \left(\frac{4H}{\hbar} \right) \; ;$$

$$L_0 = 1, \quad L_1 = 1 - \frac{4H}{\hbar}, \quad L_2 = \frac{8H^2}{\hbar^2} - \frac{8H}{\hbar} + 1 \; , ... \tag{47}$$

But for the Gaussian ground state, they all have zeros and go negative in some region.

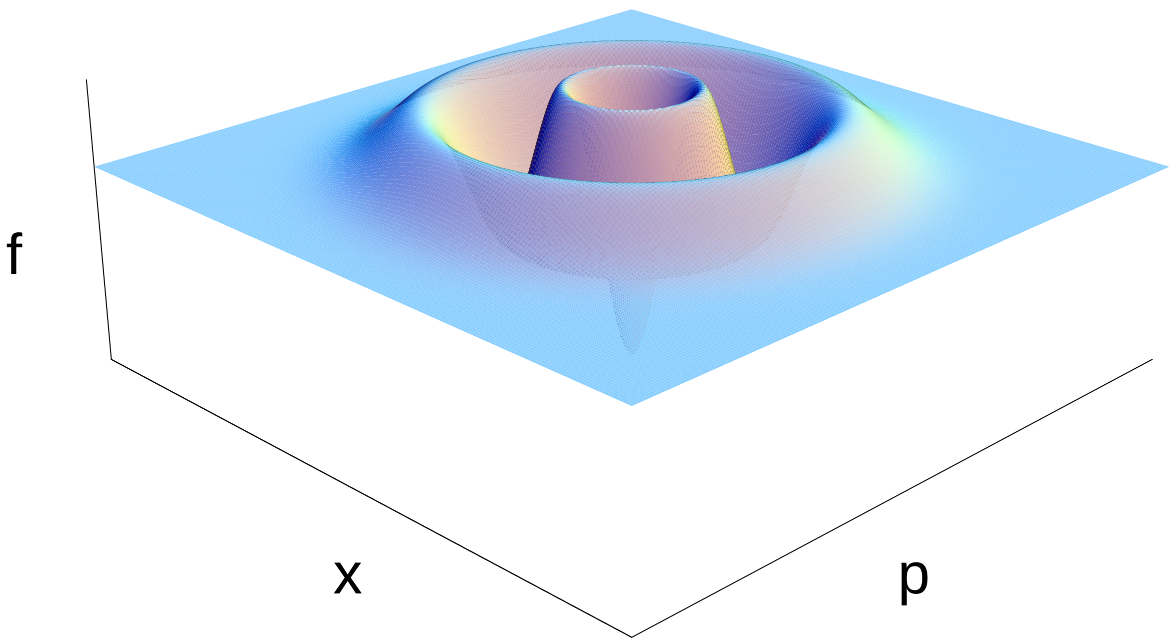


Figure 2. The oscillator WF for the 3rd excited state. Note the axial symmetry, the negative values, and the nodes.

Their sum provides a resolution of the identity[Moy49],

$$\sum_n f_n = \frac{1}{h} \; . \tag{48}$$

These Wigner functions, f_n, become spiky in the classical limit $\hbar \to 0$; e.g., the ground state Gaussian f_0 goes to a δ-function. Since, for given f_ns, $\langle x^2 + p^2 \rangle = \hbar(2n+1)$, these become "macroscopic" for very large $n = O(\hbar^{-1})$.

Note that the energy variance, the quantum fluctuation, is

$$\langle H \star H \rangle - \langle H \rangle^2 = (\langle H^2 \rangle - \langle H \rangle^2) - \frac{\hbar^2}{4} \; ,$$

vanishing for all $\star$-genstates; while the naive star-less fluctuation on the right-hand side is thus larger than that, $\hbar^2/4$, and would suggest broader dispersion, groundlessly.

(For the rest of this section, set $\hbar = 1$, for algebraic simplicity.)

Dirac's Hamiltonian factorization method for the alternate algebraic solution of this same problem carries through intact, with $\star$-multiplication now supplanting operator multiplication. That is to say,

$$H = \frac{1}{2}(x - ip) \star (x + ip) + \frac{1}{2} \ . \tag{49}$$

This motivates definition of raising and lowering functions (not operators)

$$a \equiv \frac{1}{\sqrt{2}}(x + ip), \qquad\qquad a^\dagger \equiv a^* = \frac{1}{\sqrt{2}}(x - ip), \tag{50}$$

where

$$a \star a^\dagger - a^\dagger \star a = 1 \ . \tag{51}$$

The annihilation functions $\star$-annihilate the $\star$-Fock vacuum,

$$a \star f_0 = \frac{1}{\sqrt{2}}(x + ip) \star e^{-(x^2+p^2)} = 0 \ . \tag{52}$$

Thus, the associativity of the $\star$-product permits the customary ladder spectrum generation[CFZ98]. The $\star$-genstates for $H \star f = f \star H$ are then

$$f_n = \frac{1}{n!}(a^\dagger \star)^n \ f_0 \ (\star a)^n \ . \tag{53}$$

They are manifestly real, like the Gaussian ground state, and left-right symmetric. It is easy to see they are $\star$-orthogonal for different eigenvalues. Likewise, they can be seen by the evident algebraic normal ordering to project to themselves, since the Gaussian ground state does, $f_0 \star f_0 = f_0/h$.

The corresponding coherent state WFs [HKN88,Sch88,CUZ01,Har01,DG80] are likewise analogous to the conventional formulation, amounting to this ground state with a displacement in the phase-space origin.

This type of analysis carries over well to a broader class of problems[CFZ98] with "essentially isospectral" pairs of partner potentials, connected with each other through Darboux transformations relying on Witten superpotentials W (cf. the Pöschl-Teller potential[Ant01]). It closely parallels the standard differential operator structure of the recursive technique. That is, the pairs of related potentials and corresponding $\star$-genstate Wigner functions are constructed recursively[CFZ98] through ladder operations analogous to the algebraic method outlined above for the oscillator.

Beyond such recursive potentials, examples of further simple systems where the $\star$-genvalue equations can be solved on first principles include the linear potential

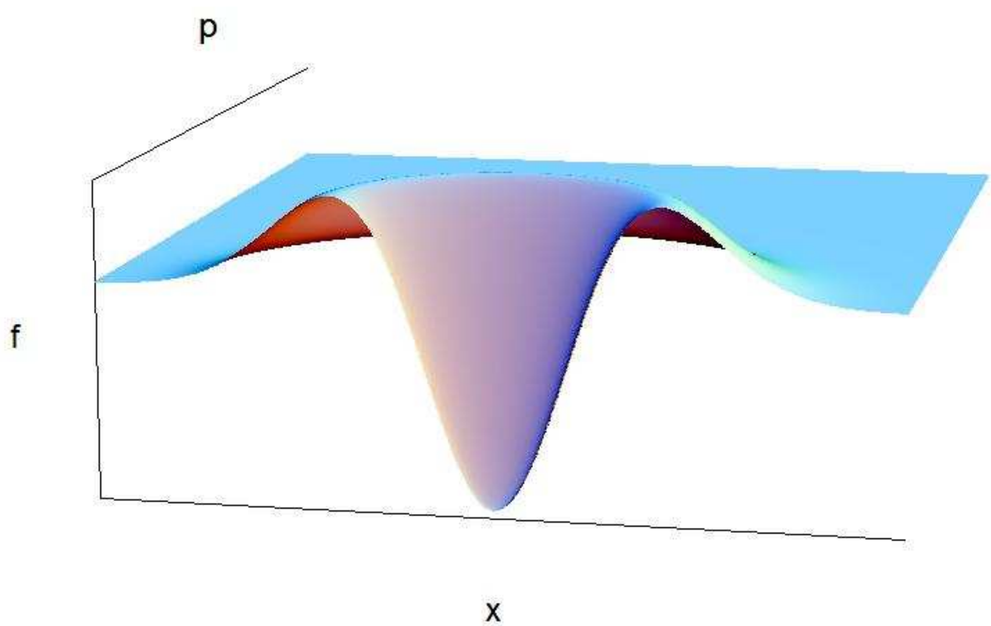


Figure 3. Section of the oscillator WF for the first excited state. Note the negative values. For this WF, $\langle z \rangle = 6$, where $z \equiv 2(x^2 + p^2)/\hbar$, as in the text. On this plot, by contrast, a "classical mechanics" oscillator of energy $3\hbar/2$ would appear as a spike at a point of $z = 6$ (beyond the ridge at $z = 3$), with its phase rotating uniformly. A uniform collection of such rotating oscillators of all phases, or a time average of one such a classical oscillator, would present as a stationary δ-function-ring at $z = 6$.

[GM80,CFZ98,TZM96], the exponential interaction Liouville potentials, and their supersymmetric Morse generalizations[CFZ98], and well-potential and δ-function limits.[KW05] (Also see [Fra00,LS82,DS82,CH86,HL99,KL94,BW10]).

Further systems may be handled through the Chebyshev-polynomial numerical techniques of ref [HMS98].

First principles phase-space solution of the Hydrogen atom is less than straightforward and complete. The reader is referred to [BFF78,Bon84,DS82,CH87] for significant partial results.

Algebraic methods of generating spectra of quantum integrable models are described in ref [CZ02].

0.7 Time Evolution

Moyal's equation (10) is formally solved by virtue of associative combinatoric operations completely analogous to Hilbert space quantum mechanics, through definition of a $\star$-unitary evolution operator, a "$\star$-exponential"[Imr67,GLS68,BFF78] ,

$$U_\star(x,p;t) = e_\star^{itH/\hbar} \equiv 1 + (it/\hbar)H(x,p) + \frac{(it/\hbar)^2}{2!} H \star H + \frac{(it/\hbar)^3}{3!} H \star H \star H + ..., \tag{54}$$

for arbitrary Hamiltonians.

The solution to Moyal's equation, given the WF at $t = 0$, then, is

$$f(x,p;t) = U_\star^{-1}(x,p;t) \star f(x,p;0) \star U_\star(x,p;t). \tag{55}$$

In general, just like any $\star$-function of H, the $\star$-exponential (54) resolves spectrally [Bon84],

$$\exp_\star\left(\frac{it}{\hbar}H\right) = \exp_\star\left(\frac{it}{\hbar}H\right) \star 1 = \exp_\star\left(\frac{it}{\hbar}H\right) \star 2\pi\hbar \sum_n f_n = 2\pi\hbar \sum_n e^{itE_n/\hbar} f_n \ . \tag{56}$$

(Of course, for $t = 0$, the obvious identity resolution is recovered.)
In turn, any particular $\star$-genfunction is projected out formally by

$$\int dt \exp_\star\left(\frac{it}{\hbar}(H - E_m)\right) = (2\pi\hbar)^2 \sum_n \delta(E_n - E_m) f_n \ \propto f_m \ , \tag{57}$$

which is manifestly seen to be a $\star$-function.

For harmonic oscillator $\star$-genfunctions, the $\star$-exponential (56) is directly seen to sum to

$$\exp_\star\left(\frac{itH}{\hbar}\right) = \left(\cos(\frac{t}{2})\right)^{-1} \exp\left(\frac{2i}{\hbar} H \tan(\frac{t}{2})\right), \tag{58}$$

which is, to say, just a Gaussian[BM49,Imr67,BFF78] in phase space[f].

[f] As an application, note that the celebrated hyperbolic tangent $\star$-composition law of Gaussians follows trivially, since these amount to $\star$-exponentials with additive time intervals, $\exp_\star(tf) \star \exp_\star(Tf) = \exp_\star((t+T)f)$,[BFF78]. That is,

$$\exp\left(-\frac{a}{\hbar}(x^2+p^2)\right) \ \star \ \exp\left(-\frac{b}{\hbar}(x^2+p^2)\right) = \frac{1}{1+ab} \exp\left(-\frac{a+b}{\hbar(1+ab)}(x^2+p^2)\right).$$

Exercise. Evaluate $e_\star^{ax} \star e_\star^{bp}$. Evaluate $\delta(x) \star \delta(p)$. Evaluate $e_\star^{ax\star p}$.

N.B. This time-evolution $\star$-exponential (56) for the harmonic oscillator may be evaluated alternatively[BFF78] without explicit knowledge of the individual $\star$-genfunctions f_n summed above. Instead, for (54), $U(H,t) \equiv \exp_\star(itH/\hbar)$, Laguerre's equation emerges again,

$$\partial_t U = \frac{i}{\hbar} H \star U = i \left(\frac{H}{\hbar} - \frac{\hbar}{4} (\partial_H + H \partial_H^2) \right) U \ ,$$

and is readily solved by (58). One may then simply read off in (56) the f_ns as the Fourier-expansion coefficients of U.

For the variables x and p, in the Heisenberg picture, the evolution equations collapse to mere *classical* trajectories for the oscillator,

$$\frac{dx}{dt} = \frac{x \star H - H \star x}{i\hbar} = \partial_p H = p \ , \tag{59}$$

$$\frac{dp}{dt} = \frac{p \star H - H \star p}{i\hbar} = -\partial_x H = -x \ , \tag{60}$$

where the concluding members of these two equations only hold for the oscillator, however.

Thus, for the oscillator,

$$x(t) = x \cos t + p \sin t, \qquad\qquad p(t) = p \cos t - x \sin t. \tag{61}$$

As a consequence, for the harmonic oscillator, the functional form of the Wigner function is preserved along classical phase-space trajectories[Gro46],

$$f(x,p;t) = f(x \cos t - p \sin t,\ p \cos t + x \sin t;\ 0). \tag{62}$$

Any oscillator WF configuration rotates uniformly on the phase plane around the origin, essentially classically, (cf. Fig. 4), even though it provides a complete quantum mechanical description[Gro46,BM49,Wig32,Les84,CZ99,ZC99].

Naturally, this rigid rotation in phase space preserves areas, and thus automatically illustrates the uncertainty principle. By contrast, in general, in the conventional formulation of quantum mechanics, this result is deprived of visualization import, or, at the very least, simplicity: upon integration in x (or p) to yield usual marginal probability densities, the rotation induces apparent complicated shape variations of the oscillating probability density profile, such as wavepacket spreading (as evident in the shadow projections on the x and p axes of Fig. 4), at least temporarily.

Only when (as is the case for coherent states[Sch88,CUZ01,HSD95,Sam00]) a Wigner function configuration has an *additional* axial $x-p$ symmetry around its *own* center, will it possess an invariant profile upon this rotation, and hence a shape-invariant oscillating probability density[ZC99].

In Dirac's interaction representation, a more complicated interaction Hamiltonian superposed on the oscillator one leads to shape changes of the WF configurations placed on the above "turntable", and serves to generalize to scalar field theory[CZ99].

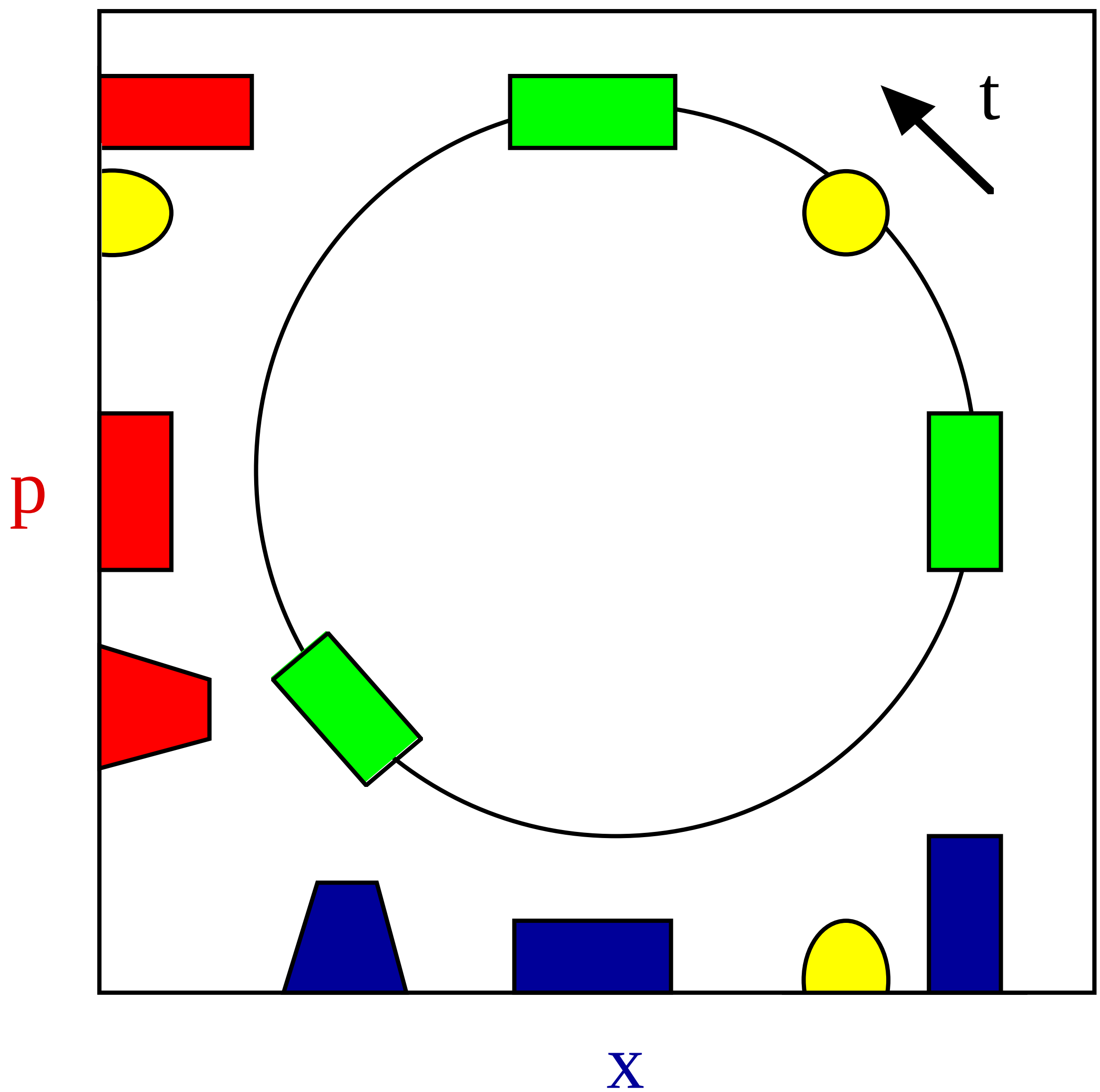


Figure 4. Time evolution of generic WF configurations driven by an oscillator Hamiltonian. The t-arrow indicates the rotation sense of x and p, and so, for fixed x and p axes, the WF shoebox configurations rotate rigidly in the opposite direction, clockwise. (The sharp angles of the WFs in the illustration are actually unphysical, and were only chosen to monitor their "spreading wavepacket" projections more conspicuously.) These x and p-projections (shadows) are meant to be intensity profiles on those axes, but are expanded on the plane to aid visualization. The circular figure portrays a coherent state (a Gaussian displaced off the origin) which projects on either axis identically at all times, thus without shape alteration of its wavepacket through time evolution.

0.8 Non-diagonal Wigner Functions

More generally, to represent all operators on phase-space in a selected basis, one looks at the Wigner-correspondents of arbitrary $|a\rangle\langle b|$, referred to as *non-diagonal WFs* [Gro46]. These enable investigation of interference phenomena and the transition amplitudes in the formulation of quantum mechanical perturbation theory [BM49,WO88,CUZ01].

Both the diagonal and the non-diagonal WFs are represented in (2), by replacing $\boldsymbol{\rho} \to |\psi_a\rangle\langle\psi_b|$,

$$\begin{aligned} f_{ba}(x,p) &\equiv \frac{1}{2\pi}\int dy\; e^{-iyp}\left\langle x+\frac{\hbar}{2}y\right|\left.\psi_a\right\rangle\;\left\langle\psi_b\right|\left.x-\frac{\hbar}{2}y\right\rangle \\ &= \frac{1}{2\pi}\int dy e^{-iyp}\psi_b^*\left(x-\frac{\hbar}{2}y\right)\psi_a\left(x+\frac{\hbar}{2}y\right) \;=\; f_{ab}^*(x,p) \\ &= \psi_a(x)\star\delta(p)\star\psi_b^*(x)\;, \end{aligned} \tag{63}$$

(NB. The *second* index is acted upon on the left.) The representation on the last line is due to [Bra94] and lends itself to a more compact and elegant proof of Lemma 0.3.

Just as pure-state diagonal WFs obey a projection condition, so too do the non-diagonals. For wave functions which are orthonormal for discrete state labels, $\int dx\,\psi_a^*(x)\psi_b(x) = \delta_{ab}$, the transition amplitude collapses to

$$\int dxdp\, f_{ab}\,(x,p) = \delta_{ab}\;. \tag{64}$$

To perform spectral operations analogous to those of Hilbert space, it is useful to note that these WFs are $\star$-orthogonal[Fai64]

$$(2\pi\hbar)\; f_{ba}\star f_{dc} = \delta_{bc} f_{da}\;, \tag{65}$$

as well as complete[Moy49] for integrable functions on phase space,

$$(2\pi\hbar)\sum_{a,b} f_{ab}\,(x_1,p_1)\; f_{ba}\,(x_2,p_2) = \delta\,(x_1-x_2)\;\delta\,(p_1-p_2)\;. \tag{66}$$

For example, for the SHO in one dimension, non-diagonal WFs are

$$f_{kn} = \frac{1}{\sqrt{n!k!}}\,(a^*\star)^n\; f_0\;(\star a)^k\,, \qquad\qquad f_0 = \frac{1}{\pi\hbar}e^{-(x^2+p^2)/\hbar}\;, \tag{67}$$

(cf. coherent states [CUZ01,Sch88,DG80]). The f_{0n} are readily identifiable[BM49,GLS68], up to a phase-space Gaussian (f_0), with the analytic Bargmann representation of wavefunctions: Note that

$$(a^*\star)^n\, f_0 = f_0\;(2a^*)^n,$$

mere functions free of operators, where $a^* = a^\dagger$, amounts to Bargmann's variable z. (Further note the limit $L_0^n = 1$ below.)

Explicitly, in terms of associated Laguerre polynomials, these are[Gro46,BM49,Fai64]

$$f_{kn} = \sqrt{\frac{k!}{n!}}\; e^{i(k-n)\ \arctan(p/x)}\, \frac{(-1)^k}{\pi\hbar} \left(\frac{x^2+p^2}{\hbar/2}\right)^{(n-k)/2} L_k^{n-k}\!\left(\frac{x^2+p^2}{\hbar/2}\right)\, e^{-\left(x^2+p^2\right)/\hbar}. \quad (68)$$

These SHO non-diagonal WFs are direct solutions to [Fai64]

$$H \star f_{kn} = E_n\, f_{kn}\ , \qquad\qquad f_{kn} \star H = E_k\, f_{kn}\ . \quad (69)$$

The resulting energy $\star$-genvalue conditions are $\left(E_n - \frac{1}{2}\right)/\hbar = n$, an integer; and $\left(E_k - \frac{1}{2}\right)/\hbar = k$, also an integer.

The general spectral theory of WFs is covered in [BFF78,FM91,Lie90,BDW99,CUZ01].

Exercise. Consider the phase-space portrayal of the simplest two-state system consisting of equal parts of oscillator ground and first-excited states. Implement the above to evaluate the corresponding rotating WF: $(f_{00}+f_{11})/2 + \Re(\exp(-it)\ f_{01})$.

0.9 Stationary Perturbation Theory

Given the spectral properties summarized, the phase-space perturbation formalism is self-contained, and it need not make reference to the parallel Hilbert-space treatment [BM49,WO88,CUZ01,SS02,MS96].

For a perturbed Hamiltonian,

$$H\,(x,p) = H_0(x,p) + \lambda\ H_1(x,p)\ , \quad (70)$$

seek a formal series solution,

$$f_n\,(x,p) = \sum_{k=0}^{\infty} \lambda^k f_n^{(k)}(x,p), \qquad\qquad E_n = \sum_{k=0}^{\infty} \lambda^k E_n^{(k)}, \quad (71)$$

of the left-right-$\star$-genvalue equations (17), $H \star f_n = E_n f_n = f_n \star H$.

Matching powers of λ in the eigenvalue equation[CUZ01],

$$E_n^{(0)} = \int dxdp\, f_n^{(0)}(x,p)\ H_0(x,p), \qquad\qquad E_n^{(1)} = \int dxdp\, f_n^{(0)}(x,p)\ H_1(x,p), \quad (72)$$

$$\begin{aligned} f_n^{(1)}\,(x,p) = &\sum_{k\neq n} \frac{f_{kn}^{(0)}\,(x,p)}{E_n^{(0)} - E_k^{(0)}} \int dXdP\, f_{nk}^{(0)}\,(X,P)\ \ H_1\,(X,P) \\ &+ \sum_{k\neq n} \frac{f_{nk}^{(0)}\,(x,p)}{E_n^{(0)} - E_k^{(0)}} \int dXdP\, f_{kn}^{(0)}\,(X,P)\ H_1\,(X,P)\ . \end{aligned} \quad (73)$$

Example. Consider all polynomial perturbations of the harmonic oscillator in a unified treatment, by choosing

$$H_1 = e^{\gamma x + \delta p} = e_\star^{\gamma x + \delta p} = \left(e^{\gamma x} \star e^{\delta p}\right)\, e^{i\gamma\delta/2} = \left(e^{\delta p} \star e^{\gamma x}\right)\, e^{-i\gamma\delta/2}\ , \quad (74)$$

to evaluate a generating function for all the first-order corrections to the energies[CUZ01],

$$E^{(1)}(s) \equiv \sum_{n=0}^{\infty} s^n E_n^{(1)} = \int dxdp \sum_{n=0}^{\infty} s^n f_n^{(0)} H_1 \ , \tag{75}$$

hence

$$E_n^{(1)} = \frac{1}{n!} \frac{d^n}{ds^n} \ E^{(1)}(s)\bigg|_{s=0} . \tag{76}$$

From the spectral resolution (56) and the explicit form of the $\star$-exponential of the oscillator Hamiltonian (58) (with $e^{it} \to s$ and $E_n^{(0)} = \left(n + \frac{1}{2}\right)\hbar$), it follows that

$$\sum_{n=0}^{\infty} s^n f_n^{(0)} = \frac{1}{\pi\hbar(1+s)} \ \exp\left(\frac{x^2+p^2}{\hbar} \ \frac{s-1}{s+1}\right) \ , \tag{77}$$

and hence

$$\begin{aligned} E^{(1)}(s) &= \frac{1}{\pi\hbar\,(1+s)} \int dxdp\, e^{\gamma x + \delta p} \exp\left(-\frac{x^2+p^2}{\hbar} \ \frac{1-s}{1+s}\right) \\ &= \frac{1}{1-s} \exp\left(\frac{\hbar}{4}\left(\gamma^2+\delta^2\right)\frac{1+s}{1-s}\right) \ . \end{aligned} \tag{78}$$

E.g., specifically,

$$\begin{aligned} E_0^{(1)} &= \exp\left(\frac{\hbar}{4}\left(\gamma^2+\delta^2\right)\right) \ , \qquad\qquad E_1^{(1)} = \left(1 + \frac{\hbar}{2}\left(\gamma^2+\delta^2\right)\right) E_0^{(1)} \ , \\ E_2^{(1)} &= \left(1 + \hbar\left(\gamma^2+\delta^2\right) + \frac{\hbar^2}{8}\left(\gamma^2+\delta^2\right)^2\right) E_0^{(1)} \ , \end{aligned} \tag{79}$$

and so on. All the first order corrections to the energies are even functions of the parameters: only even functions of x and p can contribute to first-order shifts in the harmonic oscillator energies.

First-order corrections to the WFs may be similarly calculated using generating functions for non-diagonal WFs. Higher order corrections are straightforward but tedious. Degenerate perturbation theory also admits an autonomous formulation in phase-space, equivalent to Hilbert space and path-integral treatments.

0.10 Propagators

Time evolution of general WFs beyond the above treatment is addressed at length in refs [BM49,Tak54,Ber75,GM80,CUZ01,BR93,BDR04,Wo82,Wo02,FM03,TW03].

A further application of the spectral techniques outlined is the computation of the WF time-evolution operator from the propagator for wave functions, which is given as a bilinear sum of energy eigenfunctions,

$$G(x,X;t) = \sum_a \psi_a(x)\, e^{-iE_a t/\hbar}\, \psi_a^*(X) \equiv \exp\Big(iA_{eff}(x,X;t)\Big) \ , \tag{80}$$

as it may be thought of as an exponentiated effective action. (Henceforth in this section, take $\hbar = 1$).

This leads directly to a similar bilinear double sum for the WF time-transformation kernel [Moy49],

$$T(x,p;X,P;t) = 2\pi \sum_{a,b} f_{ba}(x,p)\, e^{-i(E_a-E_b)t}\, f_{ab}(X,P)\ . \tag{81}$$

Defining a "big star" operation as a $\star$-product for the upper-case (initial) phase-space variables,

$$\bigstar \equiv e^{\frac{i\hbar}{2}(\overleftarrow{\partial}_X \overrightarrow{\partial}_P - \overleftarrow{\partial}_P \overrightarrow{\partial}_X)}\ , \tag{82}$$

it follows that

$$T(x,p;X,P;t) \bigstar f_{dc}(X,P) = \sum_b f_{bc}(x,p)\, e^{-i(E_c-E_b)t}\, f_{db}(X,P)\ , \tag{83}$$

hence, cf. (55),

$$\int dX dP\, T(x,p;X,P;t) f_{dc}(X,P) = f_{dc}(x,p) e^{-i(E_c-E_d)t} = U_\star^{-1} \star f_{dc}(x,p;0) \star U_\star = f_{dc}(x,p;t). \tag{84}$$

Example. For a free particle of unit mass in one dimension (plane wave), $H = p^2/2$, WFs propagate according to

$$\begin{aligned} & T_{free}\,(x,p;X,P;t) \\ &= \frac{1}{2\pi} \int dk \int dq\, e^{i(k-q)x}\, \delta\left(p - \frac{1}{2}(k+q)\right)\, e^{-i\left(q^2-k^2\right)t/2}\, e^{-i(k-q)X}\, \delta\left(P - \frac{1}{2}(k+q)\right) \\ &= \delta(x - X - Pt)\, \delta(p - P)\ , \end{aligned} \tag{85}$$

identifiable as "classical" free motion,

$$f(x,p;t) = f(x - pt, p; 0)\ . \tag{86}$$

The shape of any WF configuration ("wavepacket") maintains its p-profile, while shearing in x, by an amount linear in the time and p.

0.11 Canonical Transformations

Canonical transformations $(x,p) \mapsto (X(x,p), P(x,p))$ preserve the phase-space volume (area) element (again, take $\hbar = 1$) through a trivial Jacobian,

$$dXdP = dxdp\, \{X, P\}\ , \tag{87}$$

i.e., they preserve Poisson Brackets

$$\{u,v\}_{xp} \equiv \frac{\partial u}{\partial x}\frac{\partial v}{\partial p} - \frac{\partial u}{\partial p}\frac{\partial v}{\partial x}\ , \tag{88}$$

$$\{X,P\}_{xp} = 1, \qquad \{x,p\}_{XP} = 1. \tag{89}$$

Upon quantization, the c-number function Hamiltonian transforms "classically", $\mathcal{H}(X,P) \equiv H(x,p)$, like a scalar. Does the $\star$-product remain invariant under this transformation?

Yes, for *linear* canonical transformations[HKN88], but clearly *not for general canonical transformations.* Still, things can be put right, by devising general *covariant* transformation rules for the $\star$-product[CFZ98]: The WF transforms in comportance with Dirac's quantum canonical transformation theory[Dir33].

In conventional quantum mechanics, for classical canonical transformations generated by $F_{cl}(x,X)$,

$$p = \frac{\partial F_{cl}(x,X)}{\partial x} \;, \qquad\qquad P = -\frac{\partial F_{cl}(x,X)}{\partial X} \;, \tag{90}$$

the energy eigenfunctions transform in a generalization of the "representation-changing" Fourier transform[Dir33],

$$\psi_E(x) = N_E \int dX \, e^{iF(x,X)}\, \Psi_E(X) \;. \tag{91}$$

(In this expression, the generating function F may contain $\hbar$ corrections[BCT82] to the classical one, in general—but for several simple quantum mechanical systems it manages not to[CG92,DG02].) Hence[CFZ98], there is a transformation functional for WFs, $\mathcal{T}(x,p;X,P)$, such that

$$f(x,p) = \int dXdP \; \mathcal{T}(x,p;X,P) \bigstar \mathcal{F}(X,P) = \int dXdP \; \mathcal{T}(x,p;X,P) \; \mathcal{F}(X,P) \;, \tag{92}$$

where

$$\begin{aligned} &\mathcal{T}\;(x,p;X,P) \\ &= \frac{|N|^2}{2\pi}\int dY dy \;\; \exp\left(-iyp + iPY - iF^*(x-\frac{y}{2}, X-\frac{Y}{2}) + iF(x+\frac{y}{2}, X+\frac{Y}{2})\right) . \end{aligned} \tag{93}$$

Moreover, it can be shown that[CFZ98],

$$H(x,p) \star \mathcal{T}(x,p;X,P) = \mathcal{T}(x,p;X,P) \bigstar \; \mathcal{H}(X,P). \tag{94}$$

That is, if $\mathcal{F}$ satisfies a $\bigstar$−genvalue equation, then f satisfies a $\star$-genvalue equation with the same eigenvalue, and vice versa. This proves useful in constructing WFs for simple systems which can be trivialized classically through canonical transformations.

A thorough discussion of MB automorphisms may start from ref [BCW02]. (Also see [Hie82,DKM88,GR94,DV97,Hak99,KL99,DP01].)

Dynamical time evolution is also a canonical transformation [Dir33], with the generator's role played by the effective action A of the previous section, incorporating quantum corrections to both phases and normalizations; it propagates initial wave functions to those at a final time.

Example. For the linear potential, with

$$H = p^2 + x \;, \tag{95}$$

wave function evolution is determined by the propagator

$$\exp\left(iA_{lin}(x,X;t)\right)=\frac{1}{\sqrt{4\pi it}}\,\exp\left(\frac{i\,(x-X)^2}{4t}-\frac{i\,(x+X)\,t}{2}-\frac{it^3}{12}\right). \tag{96}$$

T then evaluates to

$$\begin{aligned}
&T_{lin}(\ x,p\ ;X,P;t)\\
&\quad=\ \frac{1}{2\pi}\int dY dy\ \exp\left(-iyp+iPY-iA^*_{lin}(x-\frac{y}{2},X-\frac{Y}{2};t)+iA_{lin}(x+\frac{y}{2},X+\frac{Y}{2};t)\right)\\
&\quad=\ \frac{1}{8\pi^2 t}\int dY dy\ \exp\left(-iyp+iPY-\frac{it}{2}\,(y+Y)+\frac{i}{2t}(x-X)(y-Y)\right)\\
&\quad=\ \frac{1}{2t}\,\delta\left(p+\frac{t}{2}-\frac{x-X}{2t}\right)\,\delta\left(P-\frac{t}{2}-\frac{x-X}{2t}\right)\\
&\quad=\ \delta\left(p+t-P\right)\,\delta\left(x-2tp-t^2-X\right)\\
&\quad=\ \delta\left(x-X-(p+P)\,t\right)\,\delta\left(P-p-t\right).
\end{aligned} \tag{97}$$

The δ-functions enforce exactly the classical motion for a mass$=1/2$ particle subject to a negative constant force of unit magnitude (acceleration $=-2$). Thus the WF evolves "classically" as

$$f(x,p;t)=f(x-2pt-t^2,p+t\ ;0). \tag{98}$$

NB. Time-*in*dependence follows for $f(x,p;0)$ being any function of the energy variable, since $\quad x+p^2=x-2pt-t^2+(p+t)^2$.

The evolution kernel T propagates an arbitrary WF through just[BM49]

$$f(x,p;t)=\int dXdP\ T(x,p;X,P;t)\ f(X,P;0)\ . \tag{99}$$

The underlying phase-space structure, however, is more evident if one of the wave-function propagators is given in coordinate space, and the other in momentum space. Then the path integral expressions for the two propagators can be combined into a single phase-space path integral. For every time increment, phase space is integrated over to produce the new Wigner function from its immediate ancestor. The result is

$$\begin{aligned}
&T\ (x,p;X,P;t)\\
&=\frac{1}{\pi^2}\int dx_1dp_1\int dx_2dp_2 e^{2i(x-x_1)(p-p_1)}e^{-ix_1p_1}\,\langle x_1;t|x_2;0\rangle\,\langle p_1;t|p_2;0\rangle^*\,e^{ix_2p_2}e^{-2i(X-x_2)(P-p_2)},
\end{aligned} \tag{100}$$

where $\langle x_1;t\,|x_2;0\rangle$ and $\langle p_1;t\,|p_2;0\rangle$ are the path integral expressions in coordinate space, and in momentum space. Blending these x and p path integrals gives a genuine path integral over phase space [Ber80,DK85]. For a direct connection of $U_\star$ to this integral, see ref[Sha79,Lea68,Sam00].

0.12 The Weyl Correspondence

This section summarizes the formal bridge and equivalence of phase-space quantization to the conventional operator formulation of quantum mechanics in Hilbert space. The Weyl correspondence merely provides a change of representation between phase space and Hilbert space. In itself, it does not map (commutative) classical mechanics to (non-commutative) quantum mechanics ("quantization"), as Weyl had originally hoped. But it makes the deformation map at the heart of quantization easier to grasp, now defined within a common representation, and thus more intuitive.

Weyl[Wey27] introduced an association rule mapping, invertibly, c-number phase-space functions $g(x,p)$ (called phase-space kernels) to operators $\mathfrak{G}$ in a given ordering prescription. Specifically, $p \mapsto \mathfrak{p}$, $x \mapsto \mathfrak{x}$, and, in general,

$$\mathfrak{G}(\mathfrak{x},\mathfrak{p}) = \frac{1}{(2\pi)^2}\int d\tau d\sigma dx dp\ g(x,p) \exp\Big(i\tau(\mathfrak{p}-p) + i\sigma(\mathfrak{x}-x)\Big). \tag{101}$$

The eponymous ordering prescription requires that an arbitrary operator, regarded as a power series in $\mathfrak{x}$ and $\mathfrak{p}$, be first ordered in a completely symmetrized expression in $\mathfrak{x}$ and $\mathfrak{p}$, by use of Heisenberg's commutation relations, $[\mathfrak{x},\mathfrak{p}] = i\hbar$.

A term with m powers of $\mathfrak{p}$ and n powers of $\mathfrak{x}$ is obtained from the coefficient of $\tau^m\sigma^n$ in the expansion of $(\tau\mathfrak{p}+\sigma\mathfrak{x})^{m+n}$, which serves as a generating function of Weyl-ordered polynomials[GF91]. It is evident how the map yields a Weyl-ordered operator from a polynomial phase-space kernel. It includes every possible ordering with multiplicity one, e.g.,

$$6p^2x^2 \quad \longmapsto \quad \mathfrak{p}^2\mathfrak{x}^2 + \mathfrak{x}^2\mathfrak{p}^2 + \mathfrak{p}\mathfrak{x}\mathfrak{p}\mathfrak{x} + \mathfrak{p}\mathfrak{x}^2\mathfrak{p} + \mathfrak{x}\mathfrak{p}\mathfrak{x}\mathfrak{p} + \mathfrak{x}\mathfrak{p}^2\mathfrak{x}\ . \tag{102}$$

In general[McC32],

$$p^m x^n \quad \longmapsto \quad \frac{1}{2^n}\sum_{r=0}^{n}\binom{n}{r}\mathfrak{x}^r\mathfrak{p}^m\mathfrak{x}^{n-r} = \frac{1}{2^m}\sum_{s=0}^{m}\binom{m}{s}\mathfrak{p}^s\mathfrak{x}^n\mathfrak{p}^{m-s}. \tag{103}$$

Phase-space constants map to the constant multiplying $\mathbb{1}$, the identity in Hilbert space.

In this correspondence scheme, then,

$$h\ \mathrm{Tr}\mathfrak{G} = \int dx dp\ g\ . \tag{104}$$

Conversely[Dir30,Gro46,Kub64,Lea68,HOS84], the c-number phase-space kernels $g(x,p)$ of Weyl-ordered operators $\mathfrak{G}(\mathfrak{x},\mathfrak{p})$ are specified by $\mathfrak{p} \mapsto p$, $\mathfrak{x} \mapsto x$; or, more precisely, by the "Wigner map",

$$\begin{aligned} g(x,p) &= \frac{\hbar}{2\pi}\int d\tau d\sigma\ e^{i(\tau p+\sigma x)}\mathrm{Tr}\ \left(e^{-i(\tau\mathfrak{p}+\sigma\mathfrak{x})}\mathfrak{G}\right) \\ &= \hbar\int dy\ e^{-iyp}\left\langle x+\frac{\hbar}{2}y\right|\mathfrak{G}(\mathfrak{x},\mathfrak{p})\left|x-\frac{\hbar}{2}y\right\rangle, \end{aligned} \tag{105}$$

since the above trace, in the coordinate representation, reduces to

$$\int dz\ e^{i\tau\sigma\hbar/2}\langle z|e^{-i\sigma\mathfrak{x}}e^{-i\tau\mathfrak{p}}\mathfrak{G}|z\rangle = \int dz e^{i\sigma(\tau\hbar/2-z)}\langle z-\hbar\tau|\mathfrak{G}|z\rangle. \tag{106}$$

Equivalently, the c-number integral kernel of the operator amounts to[Dir30,Bas48],
Lemma 0.7

$$\langle x|\mathfrak{G}|y\rangle = \int \frac{dp}{2\pi\hbar}\ \exp\left(ip\frac{(x-y)}{\hbar}\right)\ g\left(\frac{x+y}{2},p\right).$$

(**Exercise:** For the SHO, note the standard evolution amplitude $\langle x|\exp(-it\mathfrak{H}/\hbar)|0\rangle$, so the propagator $G(x,0;t)$, (80), follows by just inserting (58)* for g into, and evaluating this integral.)

Thus, the density matrix $|\psi_b\rangle\langle\psi_a|/h$ inserted in this expression[Moy49] yields the hermitean generalization of the Wigner function (63) encountered,

$$\begin{aligned} f_{ab}(x,p) &\equiv \frac{1}{2\pi}\int dy\ e^{-iyp}\left\langle x+\frac{\hbar}{2}y\ \middle|\ \psi_b\right\rangle\ \left\langle \psi_a\ \middle|\ x-\frac{\hbar}{2}y\right\rangle \\ &= \frac{1}{2\pi}\int dy e^{-iyp}\psi_a^*\left(x-\frac{\hbar}{2}y\right)\ \psi_b\left(x+\frac{\hbar}{2}y\right) \\ &= \frac{1}{(2\pi)^2}\int d\tau d\sigma\ \langle\psi_a|\ e^{i\tau(p-\mathfrak{p})+i\sigma(x-\mathfrak{x})}\ |\psi_b\rangle = \ f_{ba}^*(x,p)\ , \end{aligned} \tag{107}$$

where the $\psi_a(x)$s are (ortho-)normalized solutions of a Schrödinger problem. (Wigner[Wig32] mainly considered the diagonal elements of the pure-state density matrix, denoted above as $f_m \equiv f_{mm}$.)

As a consequence, matrix elements of operators, i.e., traces of them with the density matrix, are obtained through mere phase-space integrals[Moy49,Bas48],

$$\langle\psi_m|\mathfrak{G}|\psi_n\rangle = \int dxdp\ g(x,p)f_{mn}(x,p), \tag{108}$$

and thus expectation values follow for $m=n$, as utilized throughout in this overview.

Hence, above all,
Lemma 0.8

$$\langle\psi_m|\exp i(\sigma\mathfrak{x}+\tau\mathfrak{p})|\psi_m\rangle = \int dxdp\ f_m(x,p)\exp i(\sigma x+\tau p), \tag{109}$$

the celebrated *moment-generating functional*[Moy49,Bas48] of the Wigner distribution, codifying the expectation values of all moments. □

Products of Weyl-ordered operators are not necessarily Weyl-ordered, but may be easily reordered into unique Weyl-ordered operators through the degenerate Campbell-Baker-Hausdorff identity. In a study of the uniqueness of the Schrödinger representation, von Neumann[Neu31] adumbrated the composition rule of kernel functions in such operator products, appreciating that Weyl's correspondence was in fact a homomorphism. (Effectively, he arrived at the Fourier space convolution representation of the star product below.)

Finally, Groenewold[Gro46] neatly worked out in detail how the kernel functions (i.e. the Wigner transforms) f and g of two operators $\mathfrak{F}$ and $\mathfrak{G}$ must compose to yield the kernel (the Wigner map image, sometimes called the "Weyl symbol") of the product $\mathfrak{F}\ \mathfrak{G}$,

$$\mathfrak{F}\ \mathfrak{G} = \tfrac{1}{(2\pi)^4} \int d\xi d\eta d\xi' d\eta' dx' dx'' dp' dp'' f(x',p') g(x'',p'')$$
$$\times \exp i(\xi(\mathfrak{p}-p') + \eta(\mathfrak{x}-x'))\ \exp i(\xi'(\mathfrak{p}-p'') + \eta'(\mathfrak{x}-x'')) =$$

$$= \frac{1}{(2\pi)^4}\int d\xi d\eta d\xi' d\eta' dx' dx'' dp' dp'' f(x',p') g(x'',p'') \exp i\Big((\xi+\xi')\mathfrak{p} + (\eta+\eta')\mathfrak{x}\Big)$$

$$\times\ \exp i\left(-\xi p' - \eta x' - \xi' p'' - \eta' x'' + \frac{\hbar}{2}(\xi\eta' - \eta\xi')\right). \tag{110}$$

Changing integration variables to

$$\xi' \equiv \frac{2}{\hbar}(x - x'), \quad \xi \equiv \tau - \frac{2}{\hbar}(x - x'), \quad \eta' \equiv \frac{2}{\hbar}(p' - p), \quad \eta \equiv \sigma - \frac{2}{\hbar}(p' - p), \tag{111}$$

reduces the above integral to the fundamental isomorphism,

Theorem 0.1

$$\mathfrak{F}\ \mathfrak{G} = \frac{1}{(2\pi)^2}\int d\tau d\sigma dx dp\ \ \exp i\Big(\tau(\mathfrak{p}-p) + \sigma(\mathfrak{x}-x)\Big)\ (f\star g)(x,p), \tag{112}$$

where $f \star g$ is the expression (13).

□

Noncommutative operator multiplication Wigner-transforms to $\star$-multiplication.

The $\star$-product thus defines the transition from classical to quantum mechanics.

In fact, the failure of Weyl-ordered operators to close under multiplication may be stood on its head [Bra02], to *define* a Weyl-symmetrizing operator product, which is commutative and constitutes the Weyl transform of fg instead of the noncommutative $f\star g$. (For example,

$$2x\star p = 2xp + i\hbar \quad \mapsto \quad 2\mathfrak{x}\mathfrak{p} = \mathfrak{x}\mathfrak{p} + \mathfrak{p}\mathfrak{x} + i\hbar.$$

The classical piece of $2x\star p$ maps to the Weyl-symmetrization of the operator product, $2xp \mapsto \mathfrak{x}\mathfrak{p} + \mathfrak{p}\mathfrak{x}$.) One may then solve for the PB in terms of the MB, and, through the Weyl correspondence, reformulate Classical Mechanics in Hilbert space as a deformation of Quantum Mechanics, instead of the other way around [Bra02]!

Arbitrary operators $\mathfrak{G}(\mathfrak{x},\mathfrak{p})$ consisting of operators $\mathfrak{x}$ and $\mathfrak{p}$, in various orderings, but with the same classical limit, could be *imagined* rearranged by use of Heisenberg commutations to canonical completely symmetrized Weyl-ordered forms, in general with $O(\hbar)$ terms generated in the process.

Trivially, each one might then be inverse-transformed uniquely to its Weyl-correspondent c-number kernel function g in phase space. (However, in practice[Kub64], there is the above more direct Wigner transform formula (105), which bypasses any need for an actual explicit rearrangement. Since operator products amount to convolutions of such matrix-element integral kernels, $\langle x|\mathfrak{G}|y\rangle$, explicit reordering issues can be systematically avoided.)

Thus, operators differing from each other by different orderings of their $\mathfrak{x}$s and $\mathfrak{p}$s Wigner-map to kernel functions g coinciding with each other at $O(\hbar^0)$, but different at $O(\hbar)$, in general. Hence, in phase-space quantization, a survey of all alternate operator orderings in a problem with such ambiguities amounts to a survey of the "quantum correction" $O(\hbar)$ pieces of the respective kernel functions, i.e. the Wigner transforms of those operators, and their accounting is often systematized and expedited.

Choice-of-ordering problems then reduce to purely $\star$-product algebraic ones, as the resulting preferred orderings are specified through particular deformations in the c-number kernel expressions resulting from the particular solution in phase space[CZ02].

Exercise. Evaluate the $\star$-genvalues λ of $\Pi(x,p) \equiv \frac{h}{2}\delta(x)\delta(p)$. (One might think that spiky functions like this have no place in phase-space quantization, but they do: one may chek that this is but the phase-space kernel, i.e. the Wigner transform, of the parity operator[Roy77], $\int dx\ |-x\rangle\langle x| = \frac{h}{2(2\pi)^2}\int d\tau d\sigma \exp(i\tau\mathfrak{p} + i\sigma\mathfrak{x})$. So, then, what is $\Pi\star\Pi$?) Hint on $\Pi\star f = \lambda f$: For the SHO basis (47), what is $\Pi\star f_0(x,p)$? And what is $\Pi\star f_1(x,p)$? At $x = 0 = p$ for these, how does one see the necessity of the overall alternating signs in that basis?

0.13 Alternate Rules of Association

The Weyl correspondence rule (101) is not unique: there are a host of alternate *equivalent* association rules which specify corresponding representations. All these representations with equivalent formalisms are typified by characteristic quasi-distribution functions and $\star$-products, all systematically inter-convertible among themselves. They have been surveyed comparatively and organized in [Lee95,BJ84], on the basis of seminal classification work by Cohen [Coh66,Coh76]. Like different coordinate transformations, they may be favored by virtue of their different characteristic properties in varying applications.

For example, instead of the symmetric operator $\exp(i\tau\mathfrak{p} + i\sigma\mathfrak{x})$ underlying the Weyl transform, one might posit, instead [Lee95,HOS84], antistandard ordering,

$$\exp(i\tau\mathfrak{p})\exp(i\sigma\mathfrak{x}) = \exp(i\tau\mathfrak{p} + i\sigma\mathfrak{x})\ w(\tau,\sigma), \tag{113}$$

with $w = \exp(i\hbar\tau\sigma/2)$, which specifies the Kirkwood-Rihaczek prescription[Kir33]; or else standard ordering (momenta to the right), $w = \exp(-i\hbar\tau\sigma/2)$ instead on the right-hand-side of the above, for the Mehta prescription, also utilized by Moyal[Moy49,Yv46]; or their (real) average, $w = \cos(\hbar\tau\sigma/2)$ for the older Rivier prescription [Ter37]; or normal and antinormal orderings for the Glauber-Sudarshan prescriptions, generalizing to $w = \exp(\frac{\hbar}{4}(\tau^2 + \sigma^2))$ for the Husimi prescription [Hus40,Tak89] which is underlain by coherent states; or $w = \sin(\hbar\tau\sigma/2)/(\hbar\tau\sigma/2)$, for the Born-Jordan prescription; and so on.

The corresponding quasi-distribution functions in each representation can be obtained systematically as convolution transforms of each other[Coh76,Lee95,HOS84]; and, likewise, the kernel function observables are convolution "dressings" of each other, as are their $\star$-products [Dun88,AW70,Ber75].

Example. For instance, the (normalized) Husimi distribution follows from a "Gaussian smoothing" or "Gauss transform" invertible linear conversion map[WO87,Tak89,Lee95,AMP09] of the WF,

$$f_H = T(f) = \exp\left(\frac{\hbar}{4}(\partial_x^2 + \partial_p^2)\right) f \tag{114}$$
$$= \frac{1}{\pi\hbar}\int dx'dp' \exp\left(-\frac{(x'-x)^2 + (p'-p)^2)}{\hbar}\right) f(x',p'),$$

and likewise for the observables. (So, e.g., the oscillator hamiltonian now becomes $H_H = (p^2 + x^2 + \hbar)/2$.) Thus, for the *same operators* $\mathfrak{G}$, in this alternate ordering,

$$\langle\mathfrak{G}\rangle = \int dxdp\ g(x,p)\ \exp\left(-\frac{\hbar}{4}(\partial_x^2+\partial_p^2)\right)\ f_H = \int dxdp\ g_H\ e^{\hbar(\overleftarrow{\partial}_x\overrightarrow{\partial}_x + \overleftarrow{\partial}_p\overrightarrow{\partial}_p)/2}\ f_H . \tag{115}$$

That is, expectation values of observables now entail equivalence conversion dressings of the respective kernel functions and a corresponding isomorph $\star$-product [Ba79,OW81,Vor89,Tak89,Zac00],

$$\circledast = \exp\left(\frac{\hbar}{2}(\overleftarrow{\partial}_x\overrightarrow{\partial}_x + \overleftarrow{\partial}_p\overrightarrow{\partial}_p)\right) \quad \star \quad = \exp\left(\frac{\hbar}{2}(\overleftarrow{\partial}_x - i\overleftarrow{\partial}_p)(\overrightarrow{\partial}_x + i\overrightarrow{\partial}_p)\right),$$

cf. (120) below. Evidently, however, this one now *cannot be simply dropped inside integrals*, quite *unlike* the case of the WF (16).

For this reason, quantum distributions such as this Husimi distribution (which is actually [deB67,Car76,OW81,Ste80] positive semi-definite[g]) *cannot* be automatically thought of as bona-fide probability distributions, in some contrast to the WF.

This is often dramatized as the failure of the Husimi distribution f_H to yield the correct x- or p-marginal probabilities, upon integration by p or x, respectively[OW81,HOS84]. Since phase-space integrals are thus complicated by conversion dressing convolutions, they preclude direct implementation of the Schwarz inequality and the standard inequality-based moment-constraining techniques of probability theory, as well as routine completeness- and orthonormality-based functional-analytic operations.

Ignoring the above equivalence dressings and, instead, simply treating the Husimi distribution as an ordinary probability distribution in evaluating expectation values, nevertheless, results in loss of quantum information—effectively "coarse-graining" (filtering) to a semi-classical limit, and thereby increasing the relevant entropy[Bra94].

Similar caveats also apply to more recent symplectic tomographic representations [MMT96,MMM01,Leo97], which are also positive semi-definite, but also do not quite constitute conventional probability distributions.

[g] This is evident from the factorization of the constituent integrals of $f_H(0,0)$ to a complex norm squared, or, more directly, the footnote of Section (0.4) since the Gaussian is f_0 for the harmonic oscillator; and hence at *all points* in phase space.

Exercise. One may work out Moyal's inter-relations[Moy49,Coh66,Coh76] between the Weyl-ordering kernel (Wigner transform) functions and the standard-ordering correspondents; as well as the respective dressing relations between the proper $\star$-products[Lee95], in systematic analogy to the foregoing example for the Husimi prescription. The weight $w = \exp(-i\hbar\tau\sigma/2)$ mentioned dictates a dressing of kernels, $g_s = T(g) \equiv \exp(-i\hbar\partial_x\partial_p/2)\, g(x,p)$, and of $\star$-products by (120) below.

Further abstracting the Weyl-map functional of Section (0.12), for generic Hilbert-space variables $\mathfrak{z}$ and phase-space variables z, the Weyl map compacts to an integral kernel[Kub64], $\mathfrak{G}(\mathfrak{z}) = \int dz \Delta(\mathfrak{z}, z) g(z)$, and the inverse (Wigner) map to $g(z) = h\mathrm{Tr}(\overline{\Delta}(\mathfrak{z}, z)\mathfrak{G}(\mathfrak{z}))$. Here, $h\mathrm{Tr}(\Delta(\mathfrak{z}, z)\overline{\Delta}(\mathfrak{z}, z')) = \delta^2(z - z')$, $\int dz \Delta(\mathfrak{z}, z) = \int dz \overline{\Delta}(\mathfrak{z}, z) = \mathbb{1}$, and $h\mathrm{Tr}\Delta = h\mathrm{Tr}\overline{\Delta} = 1$.

The $\star$-product is thus a convolution in the integral representation, cf. (13),

Lemma 0.9

$$f \star g = \int dz' dz'' f(z') g(z'')\ h\mathrm{Tr}\Big(\overline{\Delta}(\mathfrak{z}, z)\Delta(\mathfrak{z}, z')\Delta(\mathfrak{z}, z'')\Big)\ .$$

The dressing of these functionals then presents as $\Delta_s(\mathfrak{z}, z) = T^{-1}(z)\Delta(\mathfrak{z}, z)$, so that both prescriptions yield the *same operator* $\mathfrak{G}$, when $g_s(z) = T(z)g(z)$ and $\overline{\Delta}_s = T\overline{\Delta}$.

Thus, more abstractly, the corresponding integral kernel for $\circledast$ amounts to just $h\mathrm{Tr}(T(z)\overline{\Delta}(\mathfrak{z}, z)T^{-1}(z')\Delta(\mathfrak{z}, z')T^{-1}(z'')\Delta(\mathfrak{z}, z''))$.

0.14 The Groenewold–van Hove Theorem and the Uniqueness of MBs and $\star$-products

Groenewold's correspondence principle theorem[Gro46] (to which van Hove's extension to all association rules is often attached[vH51,AB65,Ar83]) enunciates that, in general, there is *no invertible linear map* from *all functions* of phase space $f(x,p), g(x,p), \ldots$, to hermitean operators in Hilbert space $\mathfrak{Q}(f)$, $\mathfrak{Q}(g), \ldots$, such that the PB structure is preserved,

$$\mathfrak{Q}(\{f, g\}) = \frac{1}{i\hbar}\ \Big[\ \mathfrak{Q}(f), \mathfrak{Q}(g)\ \Big]\ , \tag{116}$$

as envisioned in Dirac's heuristics.[Dir25]

Instead, the Weyl correspondence map (101) from functions to ordered operators,

$$\mathfrak{W}(f) \equiv \frac{1}{(2\pi)^2} \int d\tau d\sigma dx dp\ f(x,p) \exp(i\tau(\mathfrak{p} - p) + i\sigma(\mathfrak{x} - x)), \tag{117}$$

determines the $\star$-product in (112) of Thm (0.1), $\mathfrak{W}(f \star g) = \mathfrak{W}(f)\ \mathfrak{W}(g)$, and thus the Moyal Bracket Lie algebra,

$$\mathfrak{W}(\{\!\{f, g\}\!\}) = \frac{1}{i\hbar}\Big[\mathfrak{W}(f), \mathfrak{W}(g)\Big]. \tag{118}$$

It is the MB, then, instead of the PB, which maps invertibly to the quantum commutator.

That is to say, the "deformation" involved in phase-space quantization is nontrivial: the quantum (observable) functions, in general, need not coincide with the classical ones[Gro46], and often involve $O(\hbar)$ corrections, as extensively illustrated in, e.g., refs [CZ02,DS02,CH86]; also see [Got99].

For example, as was already discussed, the Wigner transform of the square of the angular momentum $\mathfrak{L}\cdot\mathfrak{L}$ turns out to be $L^2 - 3\hbar^2/2$, significantly for the ground-state Bohr orbit [She59,DS82,DS02].

Groenewold's early celebrated *counterexample* noted that the classically vanishing PB expression

$$\{x^3, p^3\} + \frac{1}{12}\{\{p^2, x^3\}, \{x^2, p^3\}\} = 0$$

is *anomalous* in implementing Dirac's heuristic proposal to substitute commutators of $\mathfrak{Q}(x), \mathfrak{Q}(p), ...,$ for PBs upon quantization: Indeed, this substitution, or the equivalent substitution of MBs for PBs, yields a *Groenewold anomaly*, $-3\hbar^2$, for this specific expression.

Exercise. Beyond Hilbert space, in phase space, check that the standard linear operator realization $\mathfrak{V}(f) \equiv i\hbar(\partial_x f\ \partial_p - \partial_p f\ \partial_x)$ satisfies (116). But is it invertible? N.B. $\mathfrak{V}(\{x, p\}) = 0$.

An alternate abstract operator realization of the above MB Lie algebra in phase space (as opposed to the Hilbert space one, $\mathfrak{W}(f)$) linearly is[FFZ89,CFZm98]

$$\mathfrak{K}(f) = f\star\ . \tag{119}$$

Realized on a toroidal phase space, upon a formal identification $\hbar \mapsto 2\pi/N$, this realization of the MB Lie algebra leads to the Lie algebra of $SU(N)$ [FFZ89], by means of Sylvester's clock-and-shift matrices[Syl82]. For generic $\hbar$, it may be thought of as a generalization of $SU(N)$ for continuous N. This allows for taking the limit $N \to \infty$, to thus contract to the PB algebra.

Essentially (up to isomorphism), the MB algebra is the unique (Lie) one-parameter deformation (expansion) of the Poisson Bracket algebra[Vey75,BFF78,FLS76,Ar83,Fle90,deW83,BCG97,TD97], a uniqueness extending to the (associative) star product.

Isomorphism allows for dressing transformations of the variables (kernel functions and WFs, as in section 0.13 on alternate orderings), through linear maps $f \mapsto T(f)$, which leads to cohomologically equivalent star-product variants, i.e. [Ba79,Vor89,BFF78],

$$T(f \star g) = T(f) \circledast T(g). \tag{120}$$

The $\star$-MB algebra is isomorphic to the algebra of $\circledast$-MB.

Computational features of $\star$-products are addressed in refs [BFF78,Han84,RO92,Zac00,EGV89,Vo78,An97,Bra94].

0.15 Omitted Miscellany

Phase-space quantization extends in several interesting directions which are not covered in such a summary introduction.

The systematic generalization of the $\star$-product to arbitrary non-flat Poisson manifolds [Kon97], is a culmination of extensions to general symplectic and Kähler geometries[Fed94, Mor86, CGR90, Kis01], and varied symplectic contexts [Ber75, Rie89, Bor96, KL92, RT00, Xu98, CPP02, BGL01].

For further work on curved spaces, cf. ref[APW02, BF81, PT99]. For extensive reviews of mathematical issues, cf. ref[Fol89, Hor79, Wo98, AW70]. For a connection to the theory of modular forms, see ref [Raj02].

For WFs on discrete phase spaces (finite-state systems), see, among others, refs[Woo87, KP94, OBB95, ACW98, RA99, RG00, BHP02, MPS02].

Spin is treated in ref [Str57, Kut72, BGR91, VG89, AW00]; and forays into a relativistic formulation in ref[LSU02] (also see ref[CS75, Ran66]).

Inclusion of Electromagnetic fields and gauge invariance is treated in refs [Kub64, Mue99, BGR91, LF94, LF01, JVS87, ZC99, KO00]. Subtleties of Berry's phase in phase space are addressed in ref [Sam00].

Applications of the phase-space quantum picture include efficient computation of ζ-function regularization determinants[KT07].

0.16 Brief Historical Outline

The decisive contributors to the development of the formulation are Hermann Weyl (1885-1955), Eugene Wigner (1902-1995), Hilbrand Groenewold (1910-1996), and Jose Moyal (1910-1998). The bulk of the theory is implicit in Groenewold's and Moyal's seminal papers.

But this has been a fitful story of emerging connections and chains of ever sharper reformulations. Confidence in the autonomy of the formulation accreted slowly. As a result, attribution of critical milestones cannot avoid subjectivity: it cannot automatically highlight merely the *earliest* occurrence of a construct, unless that has also been conclusive enough to yield an "indefinite stay against confusion" about the logical structure of the formulation.

H Weyl (1927)[Wey27] introduces the correspondence of "Weyl-ordered" operators to phase-space (c-number) kernel functions. The correspondence is based on Weyl's formulation of the Heisenberg group, appreciated through a discrete QM application of Sylvester's (1883)[Syl82] clock and shift matrices. The correspondence is proposed as a general quantization prescription, unsuccessfully, since it fails, e.g., with angular momentum squared.

J von Neumann (1931)[Neu31], expatiates on a Fourier transform version of the $\star$-product, in a technical aside off an analysis of the uniqueness of Schrödinger's representation, based on Weyl's Heisenberg group formulation. This then effectively promotes Weyl's correspondence rule to full isomorphism between Weyl-ordered operator multiplication and $\star$-convolution of kernel functions. Nevertheless, this result is not properly appreciated in von Neumann's celebrated own book on the Foundations of QM.

E Wigner (1932)[Wig32] introduces the eponymous phase-space distribution function controlling quantum mechanical diffusive flow in phase space. It notes the negative values, and specifies the time evolution of this function and applies it to quantum statistical mechanics. (Actually, Dirac (1930)[Dir30] has already considered a formally identical construct, and an implicit Weyl correspondence, for the electron density in a multi-electron Thomas-Fermi atom, but interprets negative values as a failure of the semiclassical approximation, and crucially hesitates about the full quantum object.)

H Groenewold (1946)[Gro46], a seminal but inadequately appreciated paper, is based on Groenewold's thesis work. It achieves full understanding of the Weyl correspondence as an invertible transform, rather than as a consistent quantization rule. It articulates and recognizes the WF as the phase-space (Weyl) kernel of the density matrix. It reinvents and streamlines von Neumann's construct into the standard $\star$-product, in a systematic exploration of the isomorphism between Weyl-ordered operator products and their kernel function compositions. It thus demonstrates how Poisson Brackets contrast crucially to quantum commutators—"Groenewold's Theorem". By way of illustration, it further works out the harmonic oscillator WF.

J Moyal (1949)[Moy49] enunciates a grand synthesis: It establishes an independent formulation of quantum mechanics in phase space. It systematically studies all expectation values of Weyl-ordered operators, and identifies the Fourier transform of their moment-generating function (their characteristic function) to the Wigner Function. It further interprets the subtlety of the "negative probability" formalism and reconciles it with the uncertainty principle and the diffusion of the probability fluid. Not least, it recasts the time evolution of the Wigner Function through a deformation of the Poisson Bracket into the Moyal Bracket (the commutator of $\star$-products, i.e., the Wigner transform of the Heisenberg commutator), and thus opens up the way for a systematic study of the semiclassical limit. Before publication, Dirac contrasts this work favorably to his own ideas on functional integration, in Bohr's Festschrift[Dir45], despite private reservations and lengthy arguments with Moyal. Various subsequent scattered observations of French investigators on the statistical approach[Yv46], as well as Moyal's, are collected in J Bass (1948)[Bas48], which further stretches to hydrodynamics.

M Bartlett and J Moyal (1949) [BM49] applies this language to calculate propagators and transition probabilities for oscillators perturbed by time-dependent potentials.

T Takabayasi (1954)[Tak54] investigates the fundamental projective normalization condition for pure state Wigner functions, and exploits Groenewold's link to the conventional density matrix formulation. It further illuminates the diffusion of wavepackets.

G Baker (1958)[Bak58] (Baker's thesis paper) envisions the logical autonomy of the formulation, sustained by the projective normalization condition as a basic postulate. It resolves measurement subtleties in the correspondence principle and appreciates the significance of the anticommutator of the $\star$-product as well, thus shifting emphasis to the $\star$-product itself, over and above its commutator.

D Fairlie (1964)[Fai64] (also see refs [Kun67,Coh76,Dah83,Bas48]) explores the time-independent counterpart to Moyal's evolution equation, which involves the $\star$-product, beyond mere Moyal Bracket equations, and derives (instead of postulating) the projective orthonormality conditions for the resulting Wigner functions. These now allow for a unique and full solution of the quantum system, in principle (without any reference to the conventional Hilbert-space formulation). Autonomy of the formulation is fully recognized.

R Kubo (1964)[Kub64] elegantly reviews, in modern notation, the representation change between Hilbert space and phase space—although in ostensible ignorance of Weyl's and Groenewold's specific papers. It applies the phase-space picture to the description of electrons in a uniform magnetic field, initiating gauge-invariant formulations and pioneering "noncommutative geometry" applications to diamagnetism and the Hall effect.

N Cartwright (1976)[Car76] notes that the WF smoothed by a phase-space Gaussian as wide or wider than the minimum uncertainty packet is positive-semidefinite. Actually, the result goes further back to at least de Bruijn (1967)[deB67] and Iagolnitzer (1969)[Iag69], if not Husimi (1940)[Hus40].

M Berry (1977)[Ber77] elucidates the subtleties of the semiclassical limit, ergodicity, integrability, and the singularity structure of Wigner function evolution. Complementary results are featured in Voros (1976-78)[Vo78].

F Bayen, M Flato, C Fronsdal, A Lichnerowicz, and D Sternheimer (1978)[BFF78] analyzes systematically the deformation structure and the uniqueness of the formulation, with special emphasis on spectral theory, and consolidates it mathematically. (Also see Berezin [Ber75].) It provides explicit illustrative solutions to standard problems and utilizes influential technical tools, such as the $\star$-exponential (already known in [Imr67,GLS68]).

A Royer (1977)[Roy77] interprets WFs as the expectation value of the operators effecting reflections in phase space. (Further see refs [Kub64,Gro76,BV94].)

G García-Calderón and M Moshinsky (1980)[GM80] implements the transition from Hilbert space to phase space to extend classical propagators and canonical transformations to quantum ones in phase space. (The most conclusive work to date is ref [BCW02]. Further see [HKN88,Hie82,DKM88,CFZ98,DV97,GR94,Hak99,KL99,DP01].)

J Dahl and M Springborg (1982)[DS82] initiates a thorough treatment of the hydrogen and other simple atoms in phase space, albeit not from first principles—the WFs are evaluated in terms of Schrödinger wave-functions.

M De Wilde and P Lecomte (1983)[deW83] consolidates the deformation theory of $\star$-products and MBs on general real symplectic manifolds, analyzes their cohomology structure, and confirms the absence of obstructions.

M Hillery, R O'Connell, M Scully, and E Wigner (1984)[HOS84] has done yeoman service to the physics community as the classic introduction to phase-space quantization and the Wigner function.

Y Kim and E Wigner (1990)[KW90] is a classic pedagogical discussion of the spread of wavepackets in phase space, uncertainty-preserving transformations, coherent and squeezed states.

B Fedosov (1994)[Fed94] initiates an influential geometrical construction of the $\star$-product on all symplectic manifolds.

T Curtright, D Fairlie, and C Zachos (1998)[CFZ98] illustrates more directly the equivalence of the time-independent $\star$-genvalue problem to the Hilbert space formulation, and hence its logical autonomy; formulates Darboux isospectral systems in phase space; works out the covariant transformation rule for general nonlinear canonical transformations (with reliance on the classic work of P Dirac (1933)[Dir33]); and thus furnishes explicit solutions of practical problems on first principles, without recourse to the Hilbert space formulation. Efficient techniques for perturbation theory are based on generating functions for complete sets of Wigner functions in T Curtright, T Uematsu, and C Zachos (2001)[CUZ01]. A self-contained derivation of the uncertainty principle in phase space is given in T Curtright and C Zachos (2001)[CZ01].

M Hug, C Menke, and W Schleich (1998)[HMS98] introduce and exemplify techniques for numerical solution of $\star$-equations on a basis of Chebyshev polynomials.

BIBLIOGRAPHY

References

AW70. G Agarwal and E Wolf, *Phys Rev* **D2** (1970) 2161; *ibid* 2187, *ibid* 2206

APW02. M Alonso, G Pogosyan, and K-B Wolf, *J Math Phys* **43** (2002) 5857 [quant-ph/0205041]

AW00. J-P Amiet and S Weigert, *Phys Rev* **A63** (2000) 012102

Ant01. J-P Antoine, J-P Gazeau, P Monceau, J Klauder, K Penson, *J Math Phys* **42** (2001) 2349 [math-ph/0012044]

An97. F Antonsen, [gr-qc/9710021]

Ara95. T Arai, *J Math Phys* **36** (1995) 622-630

AB65. R Arens and D Babbitt, *J Math Phys* **6** (1965) 1071-1075

Ar83. W Arveson, *Comm Math Phys* **89** (1983) 77-102

ACW98. N Atakishiyev, S Chumakov, and K B Wolf, *J Math Phys* **39** (1998) 6247-6261

AMP09. A Athanassoulis, N Mauser, and T Paul, *Jou Math Pur App* **91** (2009) 296-338

Bak58. G Baker, *Phys Rev* **109** (1958) 2198-2206

Bak60. G Baker, I McCarthy and C Porter, *Phys Rev* **120** (1960) 254-264

BJ84. N Balazs and B Jennings, *Phys Repts* **104** (1984) 347-391

BV90. N Balazs and A Voros, *Ann Phys* (NY) **199** (1990) 123-140

Bal75. R Balescu, *Equilibrium and Nonequilibrium Statistical Mechanics* (Wiley-Interscience, New York, 1975)

BRWK99. K Banaszek, C Radzewicz, K Wódkiewicz, and J Krasinski, *Phys Rev* **A60** (1999) 674-677

BBL80. H Bartelt, K Brenner, and A Lohmann, *Opt Commun* **32** (1980) 32-38

Bar45. M Bartlett, *Proc Camb Phil Soc* **41** (1945) 71-73

Bas48. J Bass, *Rev Scientifique* **86** No 3299 (1948) 643-652;
Compt Rend Acad Sci **221** (1945) 46-49

BM49. M Bartlett and J Moyal, *Proc Camb Phil Soc* **45** (1949) 545-553

BKM03. I Bars, I Kishimoto, and Y Matsuo, *Phys Rev* **D67** (2003) 126007

BGL01. I Batalin, M Grigoriev, and S Lyakhovich, *Theor Math Phys* **128** (2001) 1109-1139 [hep-th/0101089]

BFF78. F Bayen, M Flato, C Fronsdal, A Lichnerowicz, and D Sternheimer, *Ann Phys* (NY) **111** (1978) 61-110; *ibid* 111-151; *Lett Math Phys* **1** (1977) 521-530

BF81. F Bayen and C Fronsdal, *J Math Phys* **22** (1981) 1345-1349

Ba79. F Bayen, in *Group Theoretical Methods in Physics*, W Beiglböck et al, eds, *Lecture Notes in Physics* **94** (Springer-Verlag, Heidelberg, 1979) pp 260-271

BW10. B Belchev and M Walton, *J Phys* **A43** 225206

BJY04. A Belitsky, X Ji, and F Yuan, *Phys Rev* **D69** (2004) 074014;
A Belitsky and A Radyushkin, *Phys Rept* **418** (2005) 1-387

BDR04. M Belloni, M Doncheski, and R Robinett, *Am J Phys* **72** (2004) 1183-1192

BC99. M Benedict and A Czirják, *Phys Rev* **A60** (1999) 4034-4044

BC09. G Benenti and G Casati, *Phys Rev* **E79**(2009) 025201

Ber80. F Berezin, *Sov Phys Usp* **23** (1980) 763-787

Ber75. F Berezin, *Com Math Phys* **40** (1975) 153-174; *Math USSR Izv* **8** (1974) 1109-1165

Ber77. M Berry, *Philos Trans R Soc London* **A287** (1977) 237-271;

M Berry and N Balazs, *J Phys* **A12** (1979) 625-642 ;
M Berry, "Some quantum-to-classical asymptotics", in *Les Houches Lecture Series LII (1989)*, M-J Giannoni, A Voros and J Zinn-Justin, eds, (North-Holland, Amsterdam, 1991) pp 251-304

BCG97. M Bertelson, M Cahen, and S Gutt, *Class Quant Grav* **14** (1997) A93-A107

Ber02. P Bertet et al, *Phys Rev Lett* **89** (2002) 200402

BGR91. I Bialynicki-Birula, P Górnicki, and J Rafelski, *Phys Rev* **D44** (1991) 1825-1835

BP96. B Biegel and J Plummer, *Phys Rev* **B54** (1996) 8070-8082

BHS02. M Bienert, F Haug, W Schleich, and M Raizen, *Phys Rev Lett* **89** (2002) 050403

BHP02. P Bianucci et al, *Phys Lett* **A297** (2002) 353-358

BV94. R Bishop and A Vourdas, *Phys Rev* **A50** (1994) 4488-4501

BTU93. O Bohigas, S Tomsovic, and D Ullmo, *Phys Repts* **223** (1993) 43-133

Bon84. J Gracia-Bondía, *Phys Rev* **A30** (1984) 691-697

Bor96. M Bordemann, M Brischle, C Emmrich, and S Waldmann, *Lett Math Phys* **36** (1996) 357-371

BCT82. E Braaten, T Curtright, and C Thorn, *Phys Lett* **B118** (1982) 115

BM94. A Bracken and G Melloy, *J Phys* **A27** (1994) 2197-2211

BDW99. A Bracken, H Doebner, and J Wood, *Phys Rev Lett* **83** (1999) 3758-3761;
A Bracken, D Ellinas, and J Wood *J Phys* **A36** (2003) L297-L305;
J Wood and A Bracken *J Math Phys* **46** (2005) 042103

BCW02. A Bracken, G Cassinelli, and J Wood, *J Phys* **A36** (2003) 1033-1057 [math-ph/0211001]

Bra02. A Bracken, *J Phys* **A36** (2003) L329-L335 [quant-ph/0210164]

BR93. G Braunss and D Rompf, *J Phys* **A26** (1993) 4107-4116

Bra94. G Braunss, *J Math Phys* **35** (1994) 2045-2056

BD98. D Brown and P Danielewicz, *Phys Rev* **D58** (1998) 094003

BAD96. V Bužek, G Adam, and G Drobný, *Ann Phys* (NY) **245** (1996) 37-97

CC03. A Cafarella, C Corianò, and M Guzzi, *JHEP* **11** (2003) 059

CGR90. M Cahen, S Gutt, and J Rawnsley, *J Geom Phys* **7** (1990) 45-62

CZ83. P Carruthers and F Zachariasen, *Rev Mod Phys* **55** (1983) 245-285

Car76. N Cartwright, *Physica* **83A** (1976) 210-213

CdD04. A Carvalho, R de Matos Filho, and L Davidovich, *Phys Rev* **E70** (2004) 026211

Cas91. M Casas, H Krivine, and J Martorell, *Eur J Phys* (1991) 105-111

Cas00. L Castellani, *Class Quant Grav* **17** (2000) 3377-3402 [hep-th/0005210]

CSA09. S Chaudhury, A Smith, B Anderson, S Ghose, and P Jessen, *Nature* **461** (2009) 768-771

CH87. L Chetouani and T Hammann, *J Math Phys* **28** (1987) 598-604

CH86. L Chetouani and T Hammann, *Nuov Cim* **B92** (1986) 106-120

CV98. S Chountasis and A Vourdas, *Phys Rev* **A58** (1998) 1794-1798

CL03. Y-J Chun and H-W Lee, *Ann Phys* (NY) **307** (2003) 438-451

CKTM07. W Coffey, Yu Kalmykov, S Titov, and B Mulligan, *Phys Rev* **E75** (2007) 041117; *ibid.* **E78** (2008) 031114

Coh95. L Cohen, *Time-Frequency Analysis* (Prentice Hall PTR, Englewood Cliffs, 1995)

Coh66. L Cohen, *J Math Phys* **7** (1966) 781

Coh76. L Cohen, *J Math Phys* **17** (1976) 1863

CS75. F Cooper and D Sharp, *Phys Rev* **D12** (1975) 1123-1131;

R Hakim and J Heyvaerts, *Phys Rev* **A18** (1978) 1250-1260

CPP01. H García-Compeán, J Plebanski, M Przanowski, and F Turrubiates, *Int J Mod Phys* **A16** (2001) 2533-2558

CPP02. H García-Compeán, J Plebanski, M Przanowski, and F Turrubiates, *J Phys* **A35** (2002) 4301-4320

CGB91. G Cristóbal, C Gonzalo, and J Bescós, *Advances in Electronics and Electron Physics* **80** (1991) 309-397

CG92. T Curtright and G Ghandour, in *Quantum Field Theory, Statistical Mechanics, Quantum Groups and Topology, Coral Gables 1991 Proceedings* T Curtright et al, eds (World Scientific, 1992) pp 333-344 [hep-th/9503080]

CUZ01. T Curtright, T Uematsu, and C Zachos, *J Math Phys* **42** (2001) 2396-2415 [hep-th/0011137]

CZ99. T Curtright and C Zachos, *J Phys***A32** (1999) 771-779

CFZ98. T Curtright, D Fairlie, and C Zachos, *Phys Rev* **D58** (1998) 025002

CFZm98. T Curtright, D Fairlie, and C Zachos, "Matrix Membranes and Integrability" in *Supersymmetry and Integrable Models*, Lecture Notes in Physics v 502, H Aratyn et al (eds), (Springer-Verlag, Heidelberg, 1998) pp 183-196 [hep-th/9709042]

CZ01. T Curtright and C Zachos, *Mod Phys Lett* **A16** (2001) 2381-2385

CZ02. T Curtright and C Zachos, *New J Phys* **4** (2002) 1.1-1.16 [hep-th/0205063]

DS82. J Dahl and M Springborg, *Mol Phys* **47** (1982) 1001-1019; *Phys Rev* **A36** (1988) 1050-1062; *Phys Rev* **A59** (1999) 4099-4100; *J Chem Phys* **88** (1988) 4535-4547

Dah01. J P Dahl, *Adv Quantum Chem* **39** 1-18 (2001)

DS02. J P Dahl, and W Schleich, Phys Rev **A65** (2002) 022109

Dah83. J Dahl, in *Energy Storage and Redistribution*, J Hinze, ed (Plenum Press, New York, 1983) pp 557-571

DG80. I Daubechies and A Grossmann, *J Math Phys* **21** (1980) 2080-2090;
I Daubechies, A Grossmann, and J Reignier *J Math Phys* **24** (1983) 239-254

DK85. I Daubechies and J Klauder, *J Math Phys* **26** (1985) 2239-2256

DG02. E Davis and G Ghandour, *J Phys* **35** (2002) 5875-5891 [quant-ph/9905002]

deA98. A M Ozorio de Almeida, *Phys Rept* **295** (1998) 265-342

deB67. N G de Bruijn, *"Uncertainty Principles in Fourier Analysis"* in *Inequalities*, O Shisha, ed (Academic Press, New York, 1967), pp 57-71

DeG74. S De Groot, *La Transformation de Weyl et la fonction de Wigner* (Presses de l' Université de Montreal, 1974)

Dek77. H Dekker, *Phys Rev* **A16** (1977) 2126-2134

DBB02. L Demeio, L Barletti, A Bertoni, P Bordone, and C Jacoboni, *Physica* **B314** (2002) 104-107

DV97. T Dereli and A Vercin, J Math Phys **38** (1997) 5515-5530 [quant-ph/9707040]

deW83. M de Wilde and P Lecomte, *Lett Math Phys* **7** (1983) 487

DP01. N Dias and J Prata, *J Math Phys* **42** (2001) 5565-5579

DO85. R Dickman and R O'Connell, *Phys Rev* **B32** (1985) 471-473

Dir25. P Dirac, *Proc R Soc London* **A109** (1925) 642-653

Dir30. P Dirac, *Proc Camb Phil Soc* **26** (1930) 376-385

Dir33. P Dirac, *Phys Z Sowjetunion* **3** (1933) 64-72

Dir45. P A M Dirac, *Rev Mod Phys* **17** (1945) 195-199

DKM88. R Dirl, P Kasperkovitz and M Moshinsky, *J Phys* **A21** (1988) 1835-1846

Dit90. J Dito, *Lett Math Phys* **20** (1990) 125-134; *J Math Phys* **33** (1992) 791-801

DM86. V Dodonov and V Man'ko, *Physica* **137A** (1986) 306-316

DN01. M Douglas and N Nekrasov, *Rev Mod Phys* **73** (2001) 977-1029;
R Szabo, *Phys Rept* **378** (2003) 207-299

DHS00. D Dubin, M Hennings, and T Smith, *Mathematical Aspects of Weyl Quantization and Phase* (WS, Singapore, 2000)

Dun95. T Dunne et al, *Phys Rev Lett* **74** (1995) 884-887

Dun88. G Dunne, J Phys **A21** (1988) 2321-2335

EGV89. R Estrada, J Gracia-Bondía, and J Várilly, *J Math Phys* **30** (1989) 2789-2796

Fai64. D Fairlie, *Proc Camb Phil Soc* **60** (1964) 581-586

FFZ89. D Fairlie and C Zachos, *Phys Lett* **B224** (1989) 101-107;
D Fairlie, P Fletcher and C Zachos, *J Math Phys* **31** (1990) 1088-1094

FM91. D Fairlie and C Manogue, *J Phys* **A24** (1991) 3807-3815

Fan03. A Fannjiang, *Com Math Phys* **254** (2005) 289-322 [math-ph/0304024]

Fan57. U Fano, *Rev Mod Phys* **29** (1957) 74-93

FBA96. A Farini, S Boccaletti, and F Arecchi, *Phys Rev* **E53** (1996) 4447-4450

FZ01. A Fedorova and M Zeitlin, in *PAC2001 Proceedings*, P Lucas and S Webber, eds, (IEEE, Piscataway, NJ, 2001) 1814-1816 [physics/0106005];
A Fedorova and M Zeitlin, *18th Advanced ICFA Beam Dynamics Workshop on Quantum Aspects of Beam Physics: Capri, 2000* P Chen, ed, (World Scientific, River Edge, NJ, 2002) 539-550 [physics/0101006]

Fed94. B Fedosov, *J Diff Geom* **40** (1994) 213-238

Fey87. R Feynman, "Negative Probability" in *Essays in Honor of David Bohm*, B Hiley and F Peat, eds, (Routledge and Kegan Paul, London, 1987) 235-248

FM03. S Filippas and G Makrakis *Multiscale Mod Simul* **1** (2003) 674-710

Fil96. T Filk, *Phys Lett* **B376** (1996) 53-58

FLM98. W Fischer, H Leschke, and P Müller, and P Müller, *Annalen Phys* **7** (1998) 59-100 [quant-ph/9807065]; *Phys Rev Lett* **73** (1994) 1578-1581

Fla99. P Flandrin, *Time-Frequency/Time-scale Analysis (Wavelet Analysis and Its Applications)*, (Academic Press, San Diego, 1999)

FLS76. M Flato, A Lichnerowicz, and D Sternheimer, *J Math Phys* **17** (1976) 1754

Fle90. P Fletcher, *Phys Lett* **B248** (1990) 323-328

Fol89. G Folland *Harmonic Analysis in Phase Space* (Princeton University Press, Princeton, 1989)

Fra00. A Frank, A Rivera, and K Wolf, *Phys Rev* **A61** (2000) 054102

Fre87. W Frensley, *Phys Rev* **B36** (1987) 1570-1578

FMS00. O Friesch, I Marzoli, and W Schleich, *New J Phys* **2** (2000) 4.1-4.11

Gad95. M Gadella, *Fortschr Phys* **43** (1995) 3, 229-264

GM80. G García-Calderón and M Moshinsky, *J Phys* **A13** (1980) L185

GF91. I Gelfand and D Fairlie, Commun Math Phys **136** (1991) 487-500

GH93. M Gell-Mann and Hartle, *Phys Rev* **D47** (1993) 3345-3382

GB03. J Gong and P Brumer, *Phys Rev* **A68** (2003) 062103

GLL10. M Gorbunov, K Lauritsen, and S Leroy, *Radio Sci* **45** (2010) RS6011

Got99. M Gotay, *J Math Phys* **40** (1999) 2107-2116

GR94. E Gozzi and M Reuter, *Int J Mod Phys* **A9** (1994) 5801-5820

GHSS05. B Greenbaum, S Habib, K Shizume, and B Sundaram, *Chaos* **15** (2005) 033302

Gro01. K Gröchenig, *Foundations of Time-Frequency Analysis* (Birkhäuser, Boston, 2001)
Gro46. H Groenewold, *Physica* **12** (1946) 405-460
GLS68. A Grossmann, G Loupias and E Stein, *Ann Inst Fourier* **18** (1968) 343-368.
Gro76. A Grossmann, *Comm Math Phys* **48** (1976) 191-194
Haa10. F Haake, *Quantum Signatures of Chaos* (Springer Series in Synergetics, **54**, Springer, 2010)
Hab90. S Habib, *Phys Rev* **D42** (1990) 2566-2576;
S Habib and R Laflamme, *Phys Rev* **D42** (1990) 4056-4065
Hak99. T Hakioğlu, *J Phys* **A32** (1999) 4111-4130; T Hakioğlu and A Dragt, *J Phys* **A34** (2002) 6603-6615; T Hakioğlu, A Teğmen, and B Demircioğlu, Phys Lett **A360** (2007) 501-506; T Dereli, T Hakioğlu, and A Teğmen, Int Jou Mod Phys **A24** (2009) 45-73-4587.
HY96. J Halliwell and T Yu, *Phys Rev* **D53** (1996) 2012-2019
HKN88. D Han, Y Kim, and M Noz, *Phys Rev* **A37** (1988) 807-814;
Y Kim and E Wigner, *ibid* **A38** (1988) 1159-1167; *ibid* **A36** (1987) 1293-1297
Han84. F Hansen, *Rept Math Phys* **19** (1984) 361-381
Har01. J Harvey, "Komaba Lectures on Noncommutative Solitons and D-branes" [hep-th/0102076]
HS02. A Hatzinikitas and A Smyrnakis, *J Math Phys* **A43** (2002) 113-125
Hel76. E Heller, *J Chem Phys* **65** (1976) 1289-1298;
ibid **67** (1977) 3339-3351
HSD95. M Hennings, T Smith, and D Dubin, *J Phys* **A28** (1995) 6779-6807; *ibid* 6809-6856
Hie82. J Hietarinta, *Phys Rev* **D25** (1982) 2103-2117
Hie84. J Hietarinta, *J Math Phys* **25** (1984) 1833-1840.
HOS84. M Hillery, R O'Connell, M Scully, and E Wigner, *Phys Repts* **106** (1984) 121-167
HH02. A Hirschfeld and P Henselder, *Am J Phys* **70** (2002) 537-547
HP03. M Horvat and T Prosen, *J Phys* **A36** (2003) 4015-4034
Hor79. L Hörmander, *Comm Pure App Math* **32** (1979) 359-443
HL99. X-G Hu and Q-S Li, *J Phys* **A32** (1999) 139-146
Hud74. R Hudson, *Rep Math Phys* **6** (1974) 249-252
HMS98. M Hug, C Menke, and W Schleich, *Phys Rev* **A57** (1998) 3188-3205; *ibid* 3206-3224
HW80. J Hutchinson and R Wyatt, *Chem Phys Lett* **72** (1980) 378-384
Hus40. K Husimi, *Proc Phys Math Soc Jpn* **22** (1940) 264
Iag69. D Iagolnitzer, *J Math Phys* **10** (1969) 1241-1264
Imr67. K İmre, K Özizmir, M Rosenbaum, and P Zweifel, *J Math Phys* **8** (1967) 1097-1108
JBM03. C Jacoboni, R Brunetti, and S Monastra, *Phys Rev* **B68** (2003) 125205;
C Jacoboni et al, *Rep Prog Phys* **67** (2004) 1033-1071;
P Bordone, C Jacoboni, et al, *Phys Rev* **B59** (1999) 3060 - 3069
JS02. Y Japha and B Segev, *Phys Rev* **A65** (2002) 063411
JVS87. J Javanainen, S Varró and O Serimaa, *Phys Rev* **A35** (1987) 2791-2805;
ibid **A33** (1986) 2913-2927
JG93. K Jensen and A Ganguly, *J Appl Phys* **73** (1993) 4409-4427
JN90. J Jensen and Q Niu, *Phys Rev* **A42** (1990) 2513-2519
JY98. A Jevicki and T Yoneya, *Nucl Phys* **B535** (1998) 335
JD99. A Joshi and H-T Dung, *Mod Phys Lett* **B13** (1999) 143-152
KO00. M Karasev and T Osborn, *Jou Math Phys* **43** (2002) 756-788 [quant-ph/0002041];

J Phys **A37** (2004) 2345-2363 [quant-ph/0311053]

KP94. P Kasperkovitz and M Peev, *Ann Phys* (NY) **230** (1994) 21-51

KZZ02. Z Karkuszewski, J Zakrzewski, and W Zurek, *Phys Rev* **A65** (2002) 042113;
Z Karkuszewski, C Jarzynski, and W Zurek *Phys Rev Lett* **89** (2002) 170405

KT07. B Kaynak and T Turgut, *J Math Phys* **48** (2007) 113501

KJ99. C Kiefer and E Joos, in *Quantum Future*, P Blanchard and A Jadczyk, eds (Springer-Verlag, Berlin, 1999) pp 105-128 [quant-ph/9803052];
L Diósi and C Kiefer, *J Phys* **A35** (2002) 2675-2683;
E Joos, H Zeh, C Kiefer, D Giulini, J Kupsch, I-O Stamatescu, *Decoherence and the Appearance of a Classical World in Quantum Theory* (Springer Verlag, Heidelberg, 2003)

KL99. J-H Kim and H-W Lee, *Can J Phys* **77** (1999) 411-425

KL01. K-Y Kim and B Lee, *Phys Rev* **B64** (2001) 115304

KN91. Y Kim and M Noz, *Phase Space Picture of Quantum Mechanics*, Lecture Notes in Physics v 40 (World Scientific, Singapore, 1991)

KW90. Y Kim and E Wigner, *Am J Phys* **58** (1990) 439-448

KW87. Y Kim and E Wigner, *Phys Rev* **A36** (1987) 1293; *ibid* **A38** (1988) 1159

Kir33. J Kirkwood, *Phys Rev* **44** (1933) 31-37

Kis01. I Kishimoto, *JHEP* **0103** (2001) 025

KL92. S Klimek and A Lesniewski, *Comm Math Phys* **146** (1992) 103-122

KKFR89. N Kluksdahl, A Kriman, D Ferry, and C Ringhofer, *Phys Rev* **B39** (1989) 7720-7735

Kol96. A Kolovsky, *Phys Rev Lett* **76** (1996) 340-343

KS02. A Konechny and A Schwarz, *Phys Repts* **360** (2002) 353-465

KL94. H Konno and P Lomdahl, *Jou Phys Soc Jp* **63** (1994) 3967-3973

Kon97. M Kontsevich *Lett Math Phys* **66** (2003) [q-alg/9709040]; *ibid* **48** (1999) 35-72 [math.QA/9904055]

KW05. S Kryukov and M Walton, *Ann Phys* (NY) **317** (2005) 474-491;
Can Jou Phys **84** (2006) 557-563

Kub64. R Kubo *J Phys Soc Jp* **19** (1964) 2127-2139

Kun67. W Kundt, *Z Nat Forsch* **a22** (1967) 1333-6

KPM97. C Kurtsiefer, T Pfau and J Mlynek, *Nature* **386** (1997) 150

Kut72. J Kutzner, *Phys Lett* **A41** (1972) 475-476; *Zeit f Phys* **A259** (1973) 177-188

Les84. B Lesche, *Phys Rev* **D29** (1984) 2270-2274

Lea68. B Leaf, *J Math Phys* **9** (1968) 65-72; *ibid* **9** (1968) 769-781

LS82. H-W Lee and M Scully, *J Chem Phys* **77** (1982) 4604-4610

Lee95. H-W Lee, *Phys Repts* **259** (1995) 147-211

Lei96. D Leibfried et al, *Phys Rev Lett* **77** (1996) 4281

LPM98. D Leibfried, T Pfau, and C Monroe, *Physics Today* **51** (April 1998) 22-28

Leo97. U Leonhardt, *Measuring the Quantum State of Light* (Cambridge University Press, Cambridge, 1997)

LF01. M Levanda and V Fleurov, *Ann Phys* (NY) **292** (2001) 199-231

LF94. M Levanda and V Fleurov, *J Phys: Cond Matt* **6** (1994) 7889-7908

LSU02. B Lev, A Semenov, and C Usenko, *J Russ Laser Res* **23** (2002) 347-368 [quant-ph/0112146]

Lie90. E Lieb, *J Math Phys* **31** (1990) 594-599

Lit86. R Littlejohn, *Phys Repts* **138** (1986) 193

Lou96. P Loughlin, ed, *Special Issue on Time Frequency Analysis*, *Proceeding of the IEEE* **84** (2001) No 9
Lut96. L Lutterbach and L Davidovich, *Phys Rev Lett* **78** (1997) 2547-2550
Lvo01. A Lvovsky et al, *Phys Rev Lett* **87** (2001) 050402
Mah87. G Mahan, *Phys Repts* **145** (1987) 251
MNV08. R Maia, F Nicasio, R Vallejos, and F Toscano, *Phys Rev Lett* **100** (2008) 184102
MS95. M Mallalieu and C Stroud, *Phys Rev* **A51** (1995) 1827-1835
MM84. J Martorell and E Moya, *Ann Phys* (NY) **158** 1-30
MMT96. S Mancini, V Man'ko, and P Tombesi, *Phys Lett* **A213** (1996) 1-6
MMM01. O Man'ko, V Man'ko, and G Marmo, in *Quantum Theory and Symmetries, Krakow 2001 Proceedings*, E Kapuscik and A Horzela, eds (World Scientific, 2002) [quant-ph/0112112]
MS96. M Marinov and B Segev, *Phys Rev* **A54** (1996) 4752-4762
MMP94. P Markowich, N Mauser, and F Poupaud, *J Math Phys* **35** (1995) 1066-1094
McD88. S McDonald, *Phys Repts* **158** (1988) 337-416
McC32. N McCoy, *Proc Nat Acad Sci USA* **19** (1932) 674
MOT98. B McQuarrie, T Osborn, and G Tabisz, *Phys Rev* **A58** (1998) 2944-2960;
T Osborn, M Kondrat'eva, G Tabisz, and B McQuarrie, *J Phys* **A32** (1999) 4149-4169
MMM11. C C Meaney, R McKenzie, and G Milburn, *Phys Rev* **E83** (2011) 056202
MH97. W Mecklenbräuker and F Hlawatsch, eds, *The Wigner Distribution* (Elsevier, Amsterdam, 1997);
G Matz and F Hlawatsch, *J Math Phys* **39** (1998) 4041-4069.
MPS02. C Miquel, J P Paz, and M Saraceno, *Phys Rev* **A65** (2002) 062309
Mon94. T Monteiro, *J Phys* **A27** (1994) 787-800
Mor86. C Moreno, *Lett Math Phys* **11** (1986) 361-372
Moy49. J Moyal, *Proc Camb Phil Soc* **45** (1949) 99-124
MM94. S Mrówczyński and B Müller, *Phys Rev* **D50** (1994) 7542-7552
MLD86. W Mückenheim, G Ludwig, C Dewdney, et al, *Phys Repts* **133** (1986) 337-401
Mue99. M Müller, *J Phys* **A32** (1999) 1035-1052
Na97. H Nachbagauer, [hep-th/9703105]
NO86. F Narcowich and R O'Connell, *Phys Rev* **A34** (1986) 1-6;
F Narkowich *J Math Phys* **28** (1987) 2873-2882
Neu31. J v Neumann, *Math Ann* **104** (1931) 570-578
NH08. S Nimmrichter and K Hornberger, *Phys Rev* **A78** (2008) 023612
OW81. R O'Connell and E Wigner, *Phys Lett* **85A** (1981) 121-126
OC03. R O'Connell, *J Opt* **B5** (2003) S349-S359;
M Murakami, G Ford, and R O'Connell, *Laser Phys* **13** (2003) 180-183
OBB95. T Opatrný, V Bužek, J Bajer, and G Drobný, *Phys Rev* **A52** (1995) 2419-2428; *Phys Rev* **A53** (1996) 3822-3835
OR57. I Oppenheim and J Ross, *Phys Rev* **107** (1957) 28-32
OM95. T Osborn and F Molzahn, *Ann Phys* (NY) **241** (1995) 79-127
Pei33. R Peierls, *Z Phys* **80** (1933) 763
PT99. M Przanowski and J Tosiek *Act Phys Pol* **B30** (1999) 179-201
Pul06. M Pulvirenti, *J Math Phys* **47** (2006) 052103
QC96. S Qian and D Chen, *Joint Time-Frequency Analysis* (Prentice Hall PTR, Upper

Saddle River, NJ, 1996)
Raj83. A Rajagopal, *Phys Rev* **A27** (1983) 558-561
Raj02. SG Rajeev, in *Proceedings of the 70th Meeting of Mathematicians and Physicists at Strassbourg, June 2002*, V Turaev and T Wurzbacher, eds, [hep-th/0210179]
Ram04. J Rammer, *Quantum Transport Theory* (Frontiers in Physics, Westview Press, Boulder, 2004)
Ran66. B Rankin, *Phys Rev* **141** (1966) 1223-1230
RT00. N Reshetikhin and L Takhtajan, *Amer Math Soc Transl* **201** (2000) 257-276 [math.QA/9907171]
Rie89. M Rieffel, *Comm Math Phys* **122** (1989) 531-562
RA99. A Rivas and A O de Almeida, *Ann Phys* (NY) **276** (1999) 223-256
Rob93. S Robinson, *J Math Phys* **34** (1993) 2185-2205
RO92. C Roger and V Ovsienko, *Russ Math Surv* **47** (1992) 135-191
Roy77. A Royer, *Phys Rev* **A15** (1977) 449-450
RG00. M Ruzzi and D Galetti, *J Phys* **A33** (2000) 1065-1082;
D Galetti and A de Toledo Piza, *Physica* **149A** (1988) 267-282
Sam00. J Samson, *J Phys* **A33** (2000) 5219-5229 [quant-ph/0006021]
Sch88. W Schleich, H Walther, and J A Wheeler, *Found Phys* **18** (1988) 953-968
Sch02. W Schleich, *Quantum Optics in Phase Space* (Wiley-VCH, 2002)
Sch69. J Schipper, *Phys Rev* **184** (1969) 1283-1302
SW99. N Seiberg and E Witten, *JHEP* **9909** (1999) 032
SST00. N Seiberg, L Susskind and N Toumbas, *JHEP* **0006** (2000) 044 [hep-th/0005015]
SS02. A Sergeev and B Segev, *J Phys* **A35** (2002) 1769-1789;
B Segev, *J Opt* **B5** (2003) S381-S387
Sha79. P Sharan, Phys Rev **D20** (1979) 414-418
She59. J Shewell, *Am J Phys* **27** (1959) 16-21
Shi79. Yu Shirokov, *Sov J Part Nucl* **10** (1979) 1-18
SP81. S Shlomo and M Prakash, *Nucl Phys* **A357** (1981) 157
SM00. R Simon and N Mukunda, *J Opt Soc Am* **a17** (2000) 2440-2463
Smi93. D Smithey et al, *Phys Rev Lett* **70** (1993) 1244-1247
Sny80. J Snygg, *Am J Phys* **48** (1980) 964-970
Son09. W Son et al, *Phys Rev Lett* **102** (2009) 110404
Ste80. S Stenholm, *Eur J Phys* **1** (1980) 244-248
Str57. R Stratonovich, *Sov Phys JETP* **4** (1957) 891-898
Syl82. J Sylvester, *Johns Hopkins University Circulars* **I** (1882) 241-242; *ibid* **II** (1883) 46; *ibid* **III** (1884) 7-9. Summarized in *The Collected Mathematics Papers of James Joseph Sylvester* (Cambridge University Press, 1909) v III
Tak54. T Takabayasi, *Prog Theo Phys* **11** (1954) 341-373
Tak89. K Takahashi, *Prog Theo Phys Suppl* **98** (1989) 109-156
Tat83. V Tatarskii, *Sov Phys Usp* **26** (1983) 311
Tay01. W Taylor, *Rev Mod Phys* **73** (2001) 419 [hep-th/0101126]
Ter37. Y P Terletsky, *Zh Eksp Teor Fiz*, **7** (1937) 1290-1298;
D Rivier, *Phys Rev* **83** (1951) 862-863
TGS05. M Terraneo, B Georgeot, and D L. Shepelyansky, *Phys Rev* **E71** (2005) 066215
TW03. C Trahan and R Wyatt, *Jou Chem Phys* **119** (2003) 7017-7029
TZM96. Go Torres-Vega, A Zúñiga-Segundo, and J Morales-Guzmán, *Phys Rev* **A53** (1996)

3792-3797
TA99. F Toscano and A O de Almeida, *J Phys* **A32** (1999) 6321-6346
TD97. C Tzanakis and A Dimakis, *J Phys* **A30** (1997) 4857-4866
VG89. J Várilly and J Gracia-Bondía, *Ann Phys* (NY) **190** (1989) 107-148
Vill48. J Ville, *Câbles et Transmissions* **2** (1948) 61-74
VMdG61. J Vlieger, P Mazur, and S de Groot, *Physica* **27** (1961) 353-372; 957-960; 974-078
Vog89. K Vogel and H Risken, *Phys Rev* **A40** (1989) 2847-2849
Vor89. A Voros, *Phys Rev* **A40** (1989) 6814-6825
Vo78. A Voros, *Jou Funct Analysis* **29** (1978) 104-132; *Ann Inst H Poincaré* **24** (1976) 31-90; *ibid* **26** (1977) 343-403;
B Grammaticos and A Voros, *Ann Phys* (NY) **123** (1979) 359-380
Vey75. J Vey, *Comment Math Helv* **50** (1975) 421-454
vH51. L van Hove, *Mem Acad Roy Belgique* **26** (1951) 61-102
WO87. L Wang and R O'Connell, *Physica* **144A** (1987) 201-210
WO88. L Wang and R O'Connell, *Found Phys* **18** (1988) 1023-1033
Wer95. R Werner, [quant-ph/9504016]
Wey27. H Weyl, *Z Phys* **46** (1927) 1-33;
H Weyl, *The Theory of Groups and Quantum Mechanics* (Dover, New York, 1931)
WH99. U Wiedemann and U Heinz, *Phys Repts* **319** (1999) 145-230
Wig32. E Wigner, *Phys Rev* **40** (1932) 749-759
Wis97. H Wiseman et al, *Phys Rev* **A56** (1997) 55-75
Wok97. W Wokurek, in *Proc ICASSP '97* (Munich, 1997), pp 1435-1438
Wo98. M-W Wong, *Weyl Transforms* (Springer-Verlag, Berlin, 1998)
Wo82. C-Y Wong, *Phys Rev* **C25** (1982) 1460-1475
Wo02. C-Y Wong, *J Opt* **B5** (2003) S420-S428 [quant-ph/0210112]
Woo87. W Wootters, *Ann Phys* (NY) **176** (1987) 1-21;
K Gibbons, M Hoffman, and W Wootters, *Phys Rev* **A70** 062101
WL10. X Wu and T Liu, *J Geophys Eng* **7** (2010) 126
Xu98. P Xu, *Commun Math Phys* **197** (1998) 167-197
Yv46. J Yvon, *Compt Rend Acad Sci* **223** (1946) 347-349
Yo89. T Yoneya, *Mod Phys Lett* **A4** (1989) 1587
Zac00. C Zachos, *J Math Phys* **41** (2000) 5129-5134 [hep-th/9912238];
C Zachos, *"A Survey of Star Product Geometry"* in *Integrable Hierarchies and Modern Physical Theories*, H Aratyn and A Sorin, eds, NATO Science Series II 18 (Kluwer AP, Dordrecht, 2001), pp 423-435 [hep-th/0008010]
ZC99. C Zachos and T Curtright, *Prog Theo Phys Suppl* **135** (1999) 244-258 [hep-th/9903254]
Zal03. K Zalewski *Act Phys Pol* **B34** (2003) 3379-3388
Zdn06. P Kaprálová-Žďánská, *Phys Rev* **A73** (2006) 064703
ZP94. W Zurek and J Paz, *Phys Rev Lett* **72** (1994) 2508;
S Habib, K Shizume, and W Zurek, *Phys Rev Lett* **80** (1998) 4361-4365;
W Zurek, *Rev Mod Phys* **75** (2003) 715-775
Zu91. W Zurek, *Physics Today* **44** (October 1991) 36; *Los Alamos Science* **27** (2002) 2-25

Selected Papers

0.17 H Weyl

Z Phys **46** (1927) 1-33
Quantenmechanik und Gruppentheorie [33]

0.18 J v Neumann

Math Ann **104** (1931) 570-578
Die Eindeutigkeit der Schrödingerschen Operatoren [9]

0.19 E Wigner

Phys Rev **40** (1932) 749-759
Quantum Corrections for Thermodynamic Equilibrium [11]

0.20 H Groenewold

Physica **12** (1946) 405-460
On the Principles of Elementary Quantum Mechanics [56]

0.21 J Moyal

Proc Camb Phil Soc **45** (1949) 99-124
Quantum Mechanics as a Statistical Theory [26]

0.22 M Bartlett and J Moyal

Proc Camb Phil Soc **45** (1949) 545-553
The Exact Transition Probabilities of Quantum-mechanical Oscillators Calculated by the Phase-space Method [19]

0.23 T Takabayasi

Prog Theo Phys **11** (1954) 341-373
The Formulation of Quantum Mechanics in Terms of Ensemble in Phase Space [33]

0.24 G Baker

Phys Rev **109** (1958) 2198-2206
Formulation of Quantum Mechanics Based on the Quasi-probability Distribution Induced on Phase Space [9]

0.25 D Fairlie

Proc Camb Phil Soc **60** (1964) 581-586
The Formulation of Quantum Mechanics in Terms of Phase Space Functions [6]

0.26 N Cartwright

Physica **83A** (1976) 210-212
A Non-negative Wigner-type Distribution [3]

0.27 A Royer

Phys Rev **A15** (1977) 449-450
Wigner Function as the Expectation Value of a Parity Operator [2]

0.28 F Bayen, M Flato, C Fronsdal, A Lichnerowicz, and D Sternheimer

Ann Phys **111** (1978) 61-110; **ibid** 111-151
Deformation Theory and Quantization: I. Deformations of Symplectic Structures; II. Physical Applications [91]

0.29 G Garcia-Calderon and M Moshinsky

J Phys **A13** (1980) L185-L188
Wigner Distribution Functions and the Representation of Canonical Transformations in Quantum Mechanics [4]

0.30 J Dahl and M Springborg

Mol Phys **47** (1982) 1001-1019
Wigner's Phase Space Function and Atomic Structure. I. The Hydrogen Atom Ground State [19]

0.31 M Hillery, R O'Connell, M Scully, and E Wigner

Phys Repts **106** (1984) 121-167
Distribution Functions in Physics: Fundamentals [47]

0.32 Y Kim and E Wigner

Am J Phys **58** (1990) 439-448
Canonical Transformation in Quantum Mechanics [10]

0.33 R Feynman

in *Essays in Honor of David Bohm*, B Hiley and F Peat, eds, (Routledge and Kegan Paul, London, 1987) pp 235-248
Negative Probability [14]

0.34 M De Wilde and P B A Lecomte

Lett Math Phys **7** (1983) 487-496
Existence of Star-products and of Formal Deformations of the Poisson Lie Algebra of Arbitrary Symplectic Manifolds [10]

0.35 B V Fedosov

J Diff Geom **40** (1994) 213-238
A Simple Geometrical Construction of Deformation Quantization [26]

0.36 T Curtright, D Fairlie, and C Zachos

Phys Rev **D58** (1998) 025002
Features of Time Independent Wigner Functions [14]

0.37 T Curtright and C Zachos

Mod Phys Lett **A16** (2001) 2381-2385
Negative Probability and Uncertainty Relations [5]

0.38 T Curtright, T Uematsu, and C Zachos

J Math Phys **42** (2001) 2396-2415
Generating All Wigner Functions [20]

0.39 M Hug, C Menke, and W Schleich

Phys Rev **A57** (1998) 3188-3205; *ibid* 3206-3224
Modified Spectral Method in Phase Space: Calculation of the Wigner Function: I. Fundamentals; II. Generalizations [18+19]

0.40 Note to publisher: Total Number of reprint pages: [504]

Index